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Cl. III — Vol. III, No. 3, 1955

MATHEMATICS

On a problem of J. H. C. Whitehead concerning Abelian groups

by

A. EHRENFEUCHT

Presented by K. KURATOWSKI on December 31, 1954

This paper presents a partial solution of the following problem proposed by J. H. C. Whitehead, during his visit in Warsaw in May 1952.

Let F be a free Abelian group, H a subgroup of F. Suppose that: (W) every homomorphism h of H into the group C of integers, may be extended to a homomorphism $\bar{h}$ of F into C.

Is H necessarily a direct summand of F?

For finitely generated groups F the answer is obviously "yes". The answer for the case $\bar{F}=\aleph_0$ is given below:

1. Let G be a group, $\mathcal{H}_G$ the group of all homomorphisms of G into C (addition in $\mathcal{H}_G$ is defined as the usual addition of functions).

The following lemma is well known [3] *):

(1,1) *If F is a countable, free Abelian group, then F and $\mathcal{H}_{\mathcal{H}_F}$ are isomorphic, and for every $\Phi \in \mathcal{H}_{\mathcal{H}_F}$ there exists an $a \in F$ such that $\Phi(f)=f(a)$* for all $f \in \mathcal{H}_F$.

2. Let F and H satisfy the condition (W).

(2,1) *If $a \in F-H$, then there exists a homomorphism $f \in \mathcal{H}_F$ such that $f(H)=\{0\}$ and $f(a)\neq 0$.*

Proofs: Suppose the lemma is false. Then for every pair of homomorphisms $f_1, f_2 \in \mathcal{H}_F$ which are identical in H, we have $f_1(a)=f_2(a)$. For $g \in \mathcal{H}_H$, let $\bar{g}$ denote an arbitrary homomorphism from $\mathcal{H}_F$, which is an extension of g. For $g \in \mathcal{H}_H$ we set $\psi(g)=\bar{g}(a)$. It is easy to verify that $\psi \in \mathcal{H}_{\mathcal{H}_H}$. But H, as a subgroup of the free Abelian group F, is also a free Abelian group and, in view of (1,1), there is an element $\beta \in H$ such that $\psi(g)=g(\beta)$, for $g \in H_H$. It follows that for $f \in H_F$ we have $f(a)=f(\beta)$ and therefore $a=\beta \in H$, in spite of our assumption.

3. Suppose F is countable:

$$F=\{a_1, a_2, \ldots\}.$$

*) The generalization of this lemma for groups of higher powers follows from theorem 4 in [1].

(3,1) There is a sequence of homomorphisms $h_n \in H_F$, $n=1,2,\ldots$ such that

1^0 $h_n(H)=0$,

2^0 $h_n(a_i)=0$, for $i<n$,

3^0 if $h \in H_F$, $h(H)=0$ and $h(a_i)=0$ for $i<n$, then $h_n(a_n) \mid h(a_n)$.

Proof: Let $H_n \subset H_F$ be the set of those homomorphisms h, which satisfy 1^0 and 2^0 and moreover such that $h(a_n)>0$. If H_n is not empty, then for h_n we take a homomorphism from H_n such that $h_n(a_n)=\min_{h \in H_n} h(a_n)$.

If H_n is empty, then $h_n(F)=\{0\}$.

(3,2) If $a_n \in F-H$ and $g \in H_F$, then there exist integers $c_1,\ldots,c_n$, such that $g=c_1h_1+\ldots+c_nh_n+f$, where $f \in H_F$ and $f(a_i)=0$ for $i \leqslant n$.

The proof is obvious.

4. Now let F and H satisfy condition (W) and moreover let F be countable.

(4,1) If $a \in F-H$, then there is a k, such that $h_k(a) \neq 0$.

The proof follows from (2,1) and (3,2).

(4,2) $\bigcap_{n=1}^{\infty} h_n^{-1}(0)=\{H\}$.

The proof follows from (4,1).

(4,3) The mapping $\Omega(aH)=\langle h_1(a), h_2(a), \ldots\rangle$ maps isomorphically F/H into the countable direct sum of groups C.

The proof follows from (3,1) and (4,2).

(4,4) F/H is an Abelian free group.

The proof follows from (4,3) and the well known theorem of Kulikov ([2], p. 128).

(4,5) H is a direct summand in F.

The proof follows from (4,4) and a well known theorem on free groups ([2], p. 129).

5. We have now proved the following

THEOREM. *If F is a countable, free Abelian group, H a subgroup of F and F, H satisfy condition (W), then H is a direct summand in F* *).

This theorem is a solution of the problem of Whitehead.

I wish to thank Professor A. Mostowski for suggesting this problem to me.

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*) This theorem is true for groups F of higher power, if F/H is countable.

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MATHEMATICS

On powers

by
JAN MYCIELSKI

Presented by H. STEINHAUS on January 11, 1955

The results presented in this note concern the analytical theory of numbers, but their proofs are entirely elementary, i. e. they are obtained without the use of functions of complex variables (with the exception of formula (6), in the proof of which I make use of Mangoldt's identity $\sum_{1}^{\infty}{}' \frac{\mu(m)}{m} = 0$)*). I shall present these proofs in a more detailed work, which is being prepared for the Annales Polonici Mathematici.

For every natural number $n > 1$ we denote by $a(n)$ the least natural number k and by $b(n)$ the greatest natural number l for which $n = k^l$ [2].

The letter P, with or without an index, denotes those natural numbers which are not powers; so for numbers P the equalities $a(P) = P$ and $b(P) = 1$ are characteristic [3].

The essential results of this paper concern the sequence

$$2, 3, 5, 6, 7, 10, 11, 12, 13, 14, 15, 17, \ldots,$$

i. e. the sequence $\{P\}$, in particular its distribution in the sequence of natural numbers. These results lead to certain analogies and relations between sequence $\{P\}$ and the sequence of prime numbers $\{p\}$.

Above all, let us observe that every natural number $n > 1$ can be represented in one and only one way in the form

$$P_k \ldots P_2 P_1$$

We shall denote by $P(x)$ a function analogous to $\pi(x)$, i. e. the number of those numbers of sequence $\{P\}$, which are not greater than x.

*) μ denotes Möbius' function.

The following formulas hold:

$$P(x)=\sum_{m=1}^{\infty}{}' \mu(m)\,[\sqrt[m]{x}-1] \text{ *)}, \tag{1}$$

$$P(x)=\sum_{m=2}^{\infty}{}' \frac{(\log x)^m}{m!\,\zeta(m)}+O(\log x) \text{ **)}. \tag{2}$$

Formula (1) is easy to prove; formula (2) follows in an elementary way from (1).

Let us put

$$g(x)=\sum_{m=1}^{\infty}{}' \mu(m)\,\frac{li\sqrt[m]{x}}{m}=1+\sum_{m=1}^{\infty}{}' \frac{(\log x)^m}{m!\,m\,\zeta(m+1)}, \tag{3}$$

$$h(x)=\sum_{m=1}^{\infty}{}' \mu(m)\,(\sqrt[m]{x}-1)=\sum_{m=2}^{\infty}{}' \frac{(\log x)^m}{m!\,\zeta(m)}. \tag{4}$$

The second identity of (3) and the second identity of (4) are proved by means of the identity $\sum_{1}^{\infty}{}' \frac{\mu(m)}{m}=0$. The function $g(x)$ is due to Riemann, its second representation to Gram. Thus we have

$$\pi(x)=g(x)+0(xe^{-(\log x)^{\frac{1}{2}+\frac{1}{4}\frac{1}{2}-\varepsilon}}) \text{ ***)}; \quad P(x)=h(x)+O(\log x).$$

Further,

$$g(x)=1+\int_{1}^{x}\frac{dh(t)}{\log t}; \qquad h(x)=\int_{1}^{x}\log t\,dg(t).$$

Substituting in these formulas function π for g and function P for h we obtain the approximate equalities

$$\pi(x)\approx\sum_{P\leqslant x}{}'\frac{1}{\log P}, \quad P(x)\approx\sum_{p\leqslant x}{}'\log p.$$

We may mention that

$$\sum_{P\leqslant x}{}'\frac{1}{\log P}=g(x)+O(\log\log x).$$

More generally, we have

$$\sum_{P\leqslant x}{}'(\log P)^a=\sum_{m=1}^{\infty}{}'\frac{(\log x)^{m+a+1}}{m!(m+a+1)\zeta(m+1)}+O\left((\log x)^{a+1}+\frac{(\log x)^{a+1}-1}{a+1}\right), \tag{5}$$

*) $[x]$ denotes the integral part of x.

**) ζ denotes Riemann's function.

***) Tchudakoff's theorem [1] – the strongest form of the fundamental theorem on prime numbers.

where O is uniformly bounded for $x \in (2, \infty), a \in (-1, \infty)$; similarly we have

$$\sum_{P \leqslant x}{}' P^a = \sum_{m=1}^{\infty}{}' \mu(m) \frac{x^{a+\frac{1}{m}}-1}{ma+1} + O\left(x^a \log x + \frac{x^a-1}{a}\right), \tag{6}$$

where in the case of $a=-1/m$ the m-th term of the series should be replaced by $\mu(m)\frac{\log x}{m}$.

Let us observe that formulas (5) and (6) are generalizations of (2).

Function $P(x)$ is connected with function $b(n)$ by means of the formula

$$\sum_{n=2}^{[x]}{}' F(b(n)) = \sum_{m=1}^{\infty}{}' F(m) P(\sqrt[m]{x}); \tag{7}$$

in particular, using (1), we obtain

$$\sum_{n=2}^{[x]}{}' \sum_{d \mid b(n)}{}' f(d) = \sum_{m=1}^{\infty}{}' f(m)[\sqrt[m]{x}-1]. \tag{8}$$

Particular cases of (8) are (1) and

$$\sum_{n=2}^{[x]}{}' b(n) = \sum_{m=1}^{\infty}{}' \varphi(m)[\sqrt[m]{x}-1], \text{ *}), \quad \sum_{n=2}^{[x]}{}' \sigma_a(b(n)) = \sum_{m=1}^{\infty}{}' m^a[\sqrt[m]{x}-1] \text{ **}).$$

On the basis of identities (7), (2) and (8) we can obtain asymptotic formulas for numerous sums of the form $\Sigma \Phi(b(n))$. E. g.:

$$\sum_{n=2}^{[x]}{}' (b(n))^a = \sum_{m=[a+2]}^{\infty}{}' \frac{(\log x)^m \zeta(m-a)}{m!\,\zeta(m)} + \frac{(\log x)^{[a+1]} \log\log x}{[a+1]!\,\zeta(a+1)} + O((\log x)^{a+1}),$$

$$\sum_{n=2}^{[x]}{}' \sigma_a(b(n)) = \sum_{m=[a+2]}^{\infty}{}' \frac{(\log x)^m \zeta(m-a)}{m!} + \frac{(\log x)^{[a+1]} \log\log x}{[a+1]!} + O((\log x)^{a+1}),$$

where O is uniformly bounded for $x \in (e, \infty), a \in (0, \infty)$. The latter sum can be estimated more exactly by the formula

$$\sum_{n=2}^{[x]}{}' \sigma_a(b(n)) = \sum_{m=1}^{[y]}{}' m^a \sqrt[m]{x} + \frac{(\log x)^{a+1}}{a+1} \sum_{m=2}^{[y]}{}' \frac{1}{(\log m)^{a+1}} - \frac{y^{a+2}}{a+1} + O(y^{a+1}),$$

where the number y is the function of the argument x defined by the formula $x = y^y$ and O is uniformly bounded as before.

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*) φ denotes Euler's function.

**) $\sigma_a(k)$ denotes the sum of a-th powers of the divisors of the natural number k.

The calculation of sums of the form $\Sigma\Phi(a(n))$ is more difficult. We shall only mention that

$$\sum_{n=2}^{[x]}{}' (a(n))^a = \sum_{P\leqslant x}{}' P^a + O\left(x^{\frac{a+1}{2}}\right)$$

cf. formula (6).

Let us also note the identity

$$\sum_{n=2}^{\infty}{}' \frac{1}{n^s} \sum_{d|b(n)}{}' f(d) = \sum_{m=1}^{\infty}{}' f(m)(\zeta(ms)-1)\ *),$$

whose particular cases are

$$\sum_{P}{}' \frac{1}{P^s} = \sum_{m=1}^{\infty}{}' \mu(m)(\zeta(ms)-1), \quad \sum_{n=2}^{\infty}{}' \frac{b(n)}{n^s} = \sum_{m=1}^{\infty}{}' \varphi(m)(\zeta(ms)-1)\ **)$$

and the identity

$$1 + \sum_{P} \frac{1}{P^s-1} = \zeta(s).$$

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*) We assume that function f and number s are such that the series are absolutely convergent.

**) These and other examples are given in my paper [3].

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MATHEMATICS

On infinite games

by

JAN MYCIELSKI and A. ZIĘBA

Presented by H. STEINHAUS on January 11, 1955

1. Introduction. Let us denote by $\Delta(A,B)$ a game between two players, $\mathfrak{A}$ and $\mathfrak{B}$, defined in the following way: at the apex 1 of the dendrite Δ *) (Fig. 1) a pawn is standing. In the first move, $\mathfrak{A}$ moves the pawn from that point to a lower level, to the end-point of one of two segments, 0 or 2, starting from the previous position of the pawn. Then, similarly, $\mathfrak{A}$ on odd levels and $\mathfrak{B}$ on even levels, move the pawn infinitely many times. All descending polygonal lines of the dendrite Δ are divided into two disjoint classes, A and B, known to the players. If the route of the pawn belongs to A, $\mathfrak{A}$ has won; if the route of the pawn belongs to B, $\mathfrak{B}$ has won.

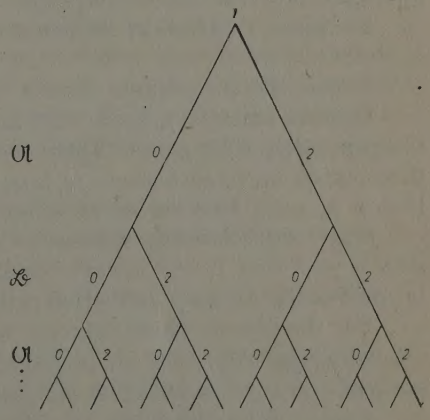

Fig. 1

The game $\Delta(A,B)$ depends, of course, on the division of the descending polygonal lines of Δ into sets A and B.

In other words $\mathfrak{A}$ chooses a function $f_{\mathfrak{A}}$, defined at point 1 and at the lower end-point of the segment from the levels of $\mathfrak{B}$ with values 0 or 2, which indicate the direction the pawn moves; independently of $\mathfrak{A}$, $\mathfrak{B}$ chooses an analogous function $f_{\mathfrak{B}}$, defined at the lower end-points of the segments from the levels of $\mathfrak{A}$. $f_{\mathfrak{A}}$ and $f_{\mathfrak{B}}$ are called *strategies*: this pair uniquely defines the play (the polygonal line), which we shall denote by $(f_{\mathfrak{A}}, f_{\mathfrak{B}})$.

*) The arithmetical definition of this dendrite is given in paper [1], p. 14.

The game is called *closed* if for one of the players there exists a strategy which makes him win, no matter what strategy is used by his opponent.

The game is called *open* if it is not closed *).

If the sets A and B in the game $\Delta(A,B)$ are such that the result of each play depends on a finite number of moves, then it is easy to see, that those numbers are mutually bounded. In that case $\Delta(A,B)$ is a finite game with perfect information ([3], p. 125) (such as chess, draughts and others). Such games are known to be closed **).

In this paper we shall prove that:

1. Although the game $\Delta(A,B)$ is intuitively a game with perfect information, yet it can be open (theorem 3).

2. The theorem on the closedness of finite games with perfect information can be generalized to some infinite games (theorems 1 and 2 and part 3 of this paper).

2. THEOREMS ***). Each play of the game $\Delta(A,B)$ is represented by means of a sequence of digits 0 and 2. Such a sequence constitutes a triadic expansion of a point of Cantor's set C — the end-point of play — polygonal line. Thus identifying the corresponding plays and points we introduce into the set of plays the topology of the set C.

THEOREM 1. *If one of the sets A and B is closed, then the game $\Delta(A,B)$ is closed.*

Proof. Let us assume that A is closed.

If there exists a $f_{\mathfrak{B}}$ such that for every $f_{\mathfrak{A}}$ we have $(f_{\mathfrak{A}},f_{\mathfrak{B}}) \in B$, then the game $\Delta(A,B)$ is closed. There remains the case, in which for every $f_{\mathfrak{B}}$ there exists an $f_{\mathfrak{A}}$ such that $(f_{\mathfrak{A}},f_{\mathfrak{B}}) \in A$. We shall prove that there exists then a $f_{\mathfrak{A}}$ such, that for every $f_{\mathfrak{B}}$ we have $(f_{\mathfrak{A}},f_{\mathfrak{B}}) \in A$.

Since A is closed it suffices to show that $\mathfrak{A}$ can move the pawn in such a way that it will always remain on some route of set A. This can be proved by an easy induction with respect to the number of moves.

For the case in which only the set B is closed, the proof is analogous.

THEOREM 2. *If the set A [the set B] is at most denumerable, then the game $\Delta(A,B)$ is closed to the advantage of player $\mathfrak{B}$ [of player $\mathfrak{A}$].*

Proof. We number all the polygonal lines of the set A [B]. It suffices for $\mathfrak{B}$ [$\mathfrak{A}$] to play in such a way that in his n-th move the pawn descends from the n-th polygonal line if it has hitherto remained on it (the idea of that proof can be found in [5]).

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*) These definitions were given for the first time in [2].

**) See [3] p. 135, corollary 6.2 or [4]. The assumption that all plays are mutually bounded is not essential in this theorem. The notion of a closed game in which a draw exists is given in part 3 of this paper (case D (A_1,A_2,A_3)).

***) Theorems 1 and 2 are effective but their generalizations, discussed in part 3 of this paper, and theorem 3 are not effective.

THEOREM 3. *There exist sets A and B such that the game $\Delta(A, B)$ is open.*

Proof. If player $\mathfrak{A}$ [player $\mathfrak{B}$] has a winning strategy $f_{\mathfrak{A}}^{(0)}$ [$f_{\mathfrak{B}}^{(0)}$], then set A [set B] must contain all the plays of type $(f_{\mathfrak{A}}^{(0)}, f_{\mathfrak{B}})$, [or type $(f_{\mathfrak{A}}, f_{\mathfrak{B}}^{(0)})$]. This is the case if, and only if, set A [set B] contains all plays of a certain dendrite of type Δ' (Fig. 2) [of type Δ'' (Fig. 3)].

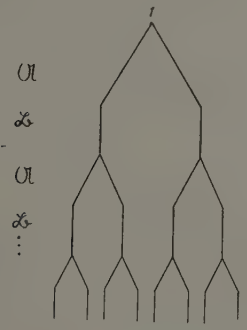

Fig. 2

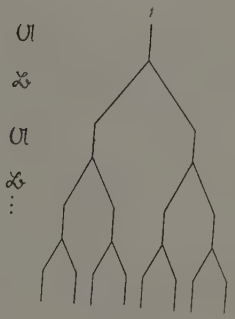

Fig. 3

The set of all plays contained in a certain dendrite of type Δ' [type Δ''] is closed and of power $\mathfrak{c}$ (it is homeomorphic with C). As we know, it can be proved by means of the axiom of choice that there exists a decomposition of set C into two disjoint sets A and B such that neither A nor B contains any closed subset of power $\mathfrak{c}$; therefore, in view of the two preceding remarks, the game $\Delta(A, B)$ corresponding to that decomposition is open.

Theorems 1, 2 and 3 suggest the following question:

(P_1) *Is the game $\Delta(A, B)$ closed if A and B are Borel sets?*

This question remains open.

3. Generalizations. The game $\Delta(A, B)$ may be generalized in two directions:

1. The dendrite Δ may be replaced by a more general figure D, such that from every position there starts a set of moves of a certain (arbitrarily great) power, not necessarily the same for all positions.

2. The plays of figure D are divided into n disjoint classes $A_1, \ldots, A_n$; player $\mathfrak{A}$ strives to minimize and player $\mathfrak{B}$ to maximize the class index of the play.

We shall call such a game $D(A_1, \ldots, A_n)$.

As before, we introduce topology to the set of plays of figure D. Namely for an arbitrary natural n we call the neighbourhood of a play Π the set of those plays which have the first n moves, i. e. the n-th position, the same as Π.

The game $D(A_1, \ldots A_n)$ will be called *closed*, if both players, $\mathfrak{A}$ and $\mathfrak{B}$, possess strategies which assure them the same class index of the play.

With regard to the games $D(A_1,\ldots A_n)$ we can prove theorems analogous to 1, 2 and 3. In particular, the theorem of Kalmár, mentioned above, on the closedness of finite games — of the type of chess — is contained in such a generalization of theorem 1. It runs as follows:

(T_1) *If for every natural i one of the sets* $\sum_{k=1}^{i} A_k$ *and* $\sum_{k=i+1}^{n} A_k$ *is closed, then the game* $D(A_1,\ldots,A_n)$ *is closed.*

The Banach-Mazur game ([5], [6]) is of type $D(A_1,A_2)$.

The games $D(A_1,\ldots,A_n)$ can also be generalized as follows *): a function f of real values is defined on the set of plays of figure D; player $\mathfrak{A}$ tries to minimize, and player $\mathfrak{B}$ to maximize the value of that function.

Such a game will be called $D(f)$.

The game $D(f)$ will be called *closed* if there exists a number x such that for every positive ε player $\mathfrak{A}$ possesses a method which ensures the inequality $f(\Pi)<x+\varepsilon$, and player $\mathfrak{B}$ possesses a method which ensures the inequality $f(\Pi)>x-\varepsilon$.

By means of theorem (T_1), case $D(A_1,A_2)$, it is easy to prove that

(T_2) *If the function f is continuous and bounded, then the game $D(f)$ is closed.*

This suggests the following question, which remains open:

(P_2) *Must the game $D(f)$ be closed if function f is a bounded Baire function?*

This question is a generalization of question (P_1) since in the space of the plays every characteristic function of a Borelian set is a Baire function.

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*) We owe this remark to Professor Ryll-Nardzewski.

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MATHEMATICS

On the theory of operator systems

by

W. SŁOWIKOWSKI

Presented by K. KURATOWSKI on January 12, 1955

In my last paper [4] I introduced the notion of an operator system. Now I wish to present a general form of the minimal ideal in operator systems and to introduce effectively the topology which was considered in the fundamental topological theorem [4]. The knowledge of [4] is here taken for granted.

Let $\mathfrak{A}=(S,X)$ be an operator system. We say that an element $x\in X$ has a *rank not greater than* $A\in S$ if there exist elements $A_1,A_2,\ldots,A_k\in S$ such that $A=A_1A_2\ldots A_k$ and $A_1(A_2\ldots(A_kx)\ldots)=0$. The element $x\in X$ belongs to $\mathfrak{I}(A)$, if there are elements $B\in S, x_1,\ldots,x_k\in X$ of the rank not greater than AB such that $x=B(x_1+x_2+\ldots+x_k)$.

THEOREM 1. *The function* $\mathfrak{I}(A)$ *is the minimal ideal of* $\mathfrak{A}$.

In a special case of the theory of distributions of finite order this minimal ideal was discussed by R. Sikorski [3].

Now let $\mathfrak{A}$ be a linear operator system such that X is a locally convex space with pseudonorms $(x|_\nu)_{\nu\in A}$. A functional $|A,x|$ defined on $S\times X$ is called a *compositional pseudonorm* of $\mathfrak{A}$ if

α) $x\|_A \overset{\text{def}}{=} |A,x|$ is a continuous pseudonorm of X for every fixed $A\in S$,

β) $|A,Bx|=|AB,x|$ for $A,B\in S$ and $x\in G(B)$.

A function $\varphi(A,x)$ defined on $S\times X$ is said to be a *compositionally linear functional* of $\mathfrak{A}$ if

γ) $f_A(x) \overset{\text{def}}{=} \varphi(A,x)$ is a continuous linear functional of X for every fixed $A\in S$;

δ) $\varphi(A,Bx)=\varphi(AB,x)$ for $A,B\in S$ and $x\in G(B)$.

We set $\varphi|_{A,\nu} \overset{\text{def}}{=} \sup_{|x|_\nu\leqslant 1} |\varphi(A,x)$ (this number may be infinite).

THEOREM 2. (a) *The functionals defined on* $S\times X$ *by the formula*

$$|A,x|_{N,\alpha} \overset{\text{def}}{=} \text{g. l. b.} \sum_{i=1}^{n} N(A_0^i)\,|A_1^i(A_2^i\ldots(A_{ki}^i,x_i)\ldots)\;\alpha(A_0^i),$$

(*where the* g. l. b. *is extended on such combinations of elements* $B, A_i^i \in S$ *and* $x_i \in X$ *that* $x = B(x_1 + x_2 + \ldots + x_n)$ *and* $A_0^i A_1^i \ldots A_{k_i}^i = AB$ *for* $i = 1, 2, \ldots n$, *and where* N *is a mapping of* S *into the set of positive integers and* α *is a mapping of* S *into the set* Λ) *are compositional pseudonorms of* $\mathfrak{A}$.

(b) *For every* $A \in S$ *and for arbitrary functions* N, α *there exist functions* M, β, *such that* $|AB, x|_{N,\alpha} \leqslant |B, x|_{M,\beta}$ *for every* $B \in S$ *and* $x \in X$.

(c) *For every compositional pseudonorm* $|A, x|$ *there exist functions* N, α *such, that* $|A, x| \leqslant |A, x|_{N,\alpha}$.

(d) *If* φ *is a compositionally linear functional of* $\mathfrak{A}$, *then we have*

$$|A, x|_{N,\alpha} = \text{l. u. b. } |\varphi(A, x)|,$$

where the l. u. b. *is extended on such compositionally linear functionals* φ *of* $\mathfrak{A}$ *that* $|\varphi|_{B,\alpha(B)} \leqslant N(B)$.

(e) *If* Φ *is the set of all compositionally linear functionals of* $\mathfrak{A}$, *then the ideal*

$$I_0(A) = \underset{x}{E}\left(\prod_{\varphi \in \Phi} \varphi(A, x) = 0\right) = \underset{x}{E}\left(\prod_{N,x} |A, x|_{N,\alpha} = 0\right)$$

coincides with the ideal considered in the fundamental topological theorem [4].

The topology in $\mathfrak{A}/\mathfrak{I}_0$ (see [4]) *is determined by pseudonorms*

$$|[z]|_{N,\alpha} \overset{\text{def}}{=} |A, x|_{N,\alpha},$$

where $[z] = H(A)h(x)$, $A \in S$, $x \in X$ *and* $\mathfrak{H} = (H, h)$ *is the corresponding natural homomorphism of* $\mathfrak{A}$ *into* $\mathfrak{A}/\mathfrak{I}_0$.

Therefore, the existence of compositionally linear functionals of $\mathfrak{A}$ is closely related with the possibility of introduction of a locally convex topology in the quotient operator system $\mathfrak{A}/\mathfrak{I}_0$. The operator system $\mathfrak{A} = (S, X)$ can be extended to locally convex algebraically closed operator system, if and only if, for every $x \in X$, $x \neq 0$ there exists an element $\varphi \in \Phi$ such, that $\varphi(I, x) \neq 0$, where I is a unit element of S.

Examples (A). Let X be the linear space of all complex-valued continuous functions of real variable. By S' we denote the semigroup of all operators $\frac{d^n}{dt^n}$ with usual operation of multiplication, by S'' — the semigroup of all quotients $s^n = \frac{1}{t^n}$ with usual operation of multiplication and by S''' we denote the semigroup of all operators $\left(a(t)\frac{d}{dt}\right)^n$ with multiplication

$$\left(a(t)\frac{d}{dt}\right)^n \left(a(t)\frac{d}{dt}\right)^m = \left(a(t)\frac{d}{dt}\right)^{n+m},$$

where $a(t)$ is a fixed element of X and $a(t) > 0$ for every $t \in (-\infty, -\infty)$.

In the operator system $\mathfrak{A}' = (S', X)$, we introduce the composition as follows. The set $G\left(\frac{d^n}{dt^n}\right)$ consists of all functions with continuous n-th derivative and $\frac{d^n x}{dt^n}$ is obviously the n-th derivative of x. In the operator-system $\mathfrak{A}'' = (S'', X)$ the function $x \in X$ belongs to $G(s^n)$ if, and only if, there exists a function $y \in X$ such that

$$x(t) = \int_0^t \left\{ \int_0^{t_n} \left\{ \dots \left\{ \int_0^{t_2} y(t_1)\, dt_1 \right\} \dots \right\} dt_{n-1} \right\} dt_n .$$

We put then $s^n x = y$. In the last operator system $\mathfrak{A}''' = (S''', X)$ we take for

$$G\left(\left(a(t)\frac{d}{dt}\right)^n\right)$$

the set of all functions $x(t)$ such that the function

$$y(t) = a(t)\frac{d}{dt}\underbrace{\left\{a(t)\frac{d}{dt}\left\{\dots\left\{a(t)\frac{dx}{dt}\right\}\dots\right\}\right\}}_{n} \in X$$

exists and is continuous.

We put then $\left(a(t)\frac{d}{dt}\right)^n x = y$.

The linear space X is a B_0 space with pseudonorms

$$|x|_n \overset{\text{def}}{=} \sup_{-n \leqslant t \leqslant +n} |x(t)| .$$

Let $C^\infty \subset X$ be the set of all infinitely differentiable functions.

The compositionally linear functionals in $\mathfrak{A}'$, $\mathfrak{A}''$ and $\mathfrak{A}'''$ have the forms:

$$\varphi_\xi\left(\frac{d^n}{dt^n}, x\right) = (-1)^n \int_0^a \frac{d^n \xi(t)}{dt^n} x(t)\, dt,$$

where $\xi \in C^\infty$ has a compact carrier

(1) $$\varphi_{\xi,a}(s^n, x) = \int_0^a \xi^{(n)}(a-t)\, x(t)\, dt,$$

where $\xi \in C^\infty$ and $\xi^{(n)}(0) = 0$ for every natural n

$$\varphi_\xi\left(\left(a(t)\frac{d}{dt}\right)^n, x\right) = (-1)^n \int_{-\infty}^{+\infty} \frac{\xi^{(n)}\left(\int_0^t \frac{d\tau}{a(\tau)}\right)}{a(t)} x(t)\, dt,$$

where $\xi \in C^\infty$ has a compact carrier contained in the rank of $a(t)$.

The following ideals, determined by the functionals (1) are the minimal ideals of $\mathfrak{A}'$, $\mathfrak{A}''$ and $\mathfrak{A}'''$ respectively:

$\mathfrak{I}'\left(\frac{d^n}{dt^n}\right)$ = the set of all polynomials of a degree $< n$,

$\mathfrak{I}''(s^n)$ = the set containing only the function identically equal to zero,

$$\mathfrak{I}'''\left(a(t)\frac{d}{dt}\right)^n = \left[C_0 + C_1 \int_0^t \frac{d\tau_1}{a(\tau_1)}\right.$$

$$+ C_2 \int_0^t \frac{1}{a(\tau_1)} \int_0^{\tau_1} \frac{d\tau_2}{a(\tau_2)} d\tau_1 + \ldots + C_{n-1} \int_0^t \left\{\frac{1}{a(\tau_1)} \int_0^{\tau_1} \left\{\frac{1}{a(\tau_2)} \cdots\right.\right.$$

$$\left.\left. \ldots \int_0^{\tau_{n-2}} \frac{d\tau_{n-1}}{a(\tau_{n-1})} \ldots\right\} d\tau_2\right\} d\tau_1\Bigg].$$

The first operator system was considered by Laurent Schwartz [2], the second may be treated as a special case of Mikusiński's theory [1]. The last example has so far not been considered.

(B) Let X be a linear space of all continuous complex-valued functions defined on the Euclidean plane. Let the semigroup S consist of all elements $\left(\frac{\partial^n}{\partial t^n}, s^m\right)$ with following multiplication

$$\left(\frac{\partial^{n_1}}{\partial t^{n_1}}, s^{m_1}\right)\left(\frac{\partial^{n_2}}{\partial t^{n_2}}, s^{m_2}\right) = \left(\frac{\partial^{n_1+n_2}}{\partial t^{n_1+n_2}}, s^{m_1+m_2}\right).$$

$G\left(\left(\frac{\partial^n}{\partial t^n}, s^m\right)\right)$ is the set of all functions $x \in X$ for which there is an element $y \in X$ such that

$$\frac{\partial^n}{\partial t^n} x(t,\tau) = \int_0^{\tau} \left\{\int_0^{\tau_m} \ldots \left\{\int_0^{\tau_2} y(t,\tau_1) d\tau_1\right\} \ldots\right\} d\tau_{m-1}\Big\} d\tau_m \qquad (*).$$

The function y in formula (*) is uniquely determined by an element $\left(\frac{\partial^n}{\partial t^n}, s^m\right)$ and the function $x \in X$. The element y is said to be the composition of $\left(\frac{\partial^n}{\partial t^n}, s^m\right)$ and x.

If we introduce in X a locally convex topology by the pseudonorms

$$|x|_n = \sup_{-n \leqslant t,\tau \leqslant +n} |x(t,\tau)|,$$

then the compositionally linear functionals of $\mathfrak{A} = (S, X)$ are

$$\varphi_{\xi,a}\left(\left(\frac{\partial_n}{\partial t^n}, s^m\right), x\right) = (-1)^n \int_{-\infty}^{+\infty} \left\{\int_0^a \frac{\partial^{n+m}\xi(p,r)}{\partial p^n \partial r^m} x(p, a-r)\, dr\right\} dp,$$

where ξ is an infinitely differentiable function of X such that for every $\tau \in (-\infty, +\infty)$ the function $f_\tau(t) \overset{\text{def}}{=} \xi(t,\tau)$ have a compact carrier and

$$\frac{\partial^n}{\partial \tau^n} \xi(t,\tau)_{\tau=0} = 0 \quad \text{for every} \quad t \in (-\infty, +\infty) \quad \text{and} \quad n = 1,2,\ldots$$

The ideal induced by these functionals is the minimal ideal of $\mathfrak{A}$ and has the form

$$\Im\left(\left(\frac{\partial^n}{\partial t^n}, s^m\right)\right) = [C_1(\tau)t^{n-1} + C_2(\tau)t^{n-2} + \ldots + C_n(\tau)],$$

where $C_i \in X$.

(C). Schwartz's idea of distributions and Mikusiński's idea of operators can both be unified in the notion of distribuo-operators. Let X be a linear space of complex-valued continuous functions $x(t,\tau)$, where $-\infty < t < +\infty$ and $0 \leqslant \tau < +\infty$. Let S be composed by all operators of the form $\left(\frac{\partial^n}{\partial t^n}, \frac{1}{a}\right)$, where n is a natural number and a is a continuous complex-valued function defined for $\tau \in <0, +\infty)$ and not identically equal to zero $\left(\text{the formal quotient } \frac{1}{a} \text{ should not be identified with the function } \frac{1}{a(t)}\right)$.

The multiplication in S is defined as follows:

$$\left(\frac{\partial^n}{\partial t^n}, \frac{1}{a}\right)\left(\frac{\partial^m}{\partial t^m}, \frac{1}{b}\right) = \left(\frac{\partial^{n+m}}{\partial t^{n+m}}, \frac{1}{a \times b}\right),$$

where $a \times b$ is a convolution of a and b, i. e.

$$a \times b(t) = \int_0^t a(t-\tau)b(\tau)\,d\tau.$$

An element $x \in X$ belongs to $G\left(\left(\frac{\partial^n}{\partial t^n}, \frac{1}{a}\right)\right)$ if there is a function $y \in X$ such that

$$\frac{\partial^n}{\partial t^n} x(t,\tau) = \int_0^\tau a(p)y(t,\tau-p)\,dp.$$

The function y is uniquely determined by the operator $\left(\frac{\partial^n}{\partial t^n}, \frac{1}{a}\right)$ and the function $x \in X$. The function y is said to be the composition of

$$\left(\frac{\partial^n}{\partial t^n}, \frac{1}{a}\right) \in S \quad \text{and} \quad x \in G\left(\left(\frac{\partial^n}{\partial t^n}, \frac{1}{a}\right)\right).$$

The minimal ideal of $\mathfrak{A}=(S,X)$ has the form

$$\Im\left(\left(\frac{\partial^n}{\partial t^n},\frac{1}{a}\right)\right)=[C_1(\tau)t^{n-1}+C_2(\tau)t^{n-2}+\ldots+C_n(\tau)]$$

where $C_i \in X$.

If we introduce in X a locally convex topology, similarly as in example (B), then this ideal cannot be determined by the compositionally linear functionals of $\mathfrak{A}$.

This is evident, because, the introduction of such a locally-convex topology is impossible in the case of Mikusiński's operators.

(D). Let X be a Hilbert space and let S be a commutative semigroup of linear operators $A \in S$ which map some dense subspaces $G(A)\subset X$ onto the whole space X. We suppose that $\bar{G}_0=X$, where

$$G_0=\prod_{A\in S} G(A^*).$$

($G(A^*)$ is the domain of the operator A^* adjoint to the operator $A \in S$).

Then every compositionally linear functional of $\mathfrak{A}=(S,X)$ have the form

$$\varphi_\xi(A,x)=(x,A^*\xi),$$

where $\xi \in G_0$, and the ideal generated by these functionals is the zero ideal.

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MATHÉMATIQUE

Sur la structure de l'ensemble engendré par les intégrales non asymptotiques des équations différentielles

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Dans cette communication seront établis quelques théorèmes relatifs à l'examen de l'allure asymptotique des intégrales d'équations différentielles par la méthode des tuyaux [1]. Les travaux antérieurs qui ont fourni à l'aide de cette méthode les conditions d'existence des intégrales asymptotiques par rapport à un tuyau ω étaient basés sur différentes notions de rétracte ([1], [2], [3], [4]). En rapport avec les remarques formulées dans [5], je vais montrer ici que, sans résoudre les équations, il est possible, sous des hypothèses bien générales, de déterminer complètement la structure topologique de l'ensemble Γ des points du tuyau $\bar{\omega}$ qui n'appartiennent pas à l'ensemble Z engendré par les intégrales asymptotiques par rapport à ω (Théorèmes 3 et 4). Ces résultats, en particulier les Théorèmes 1 et 2, seront appliqués dans une communication ultérieure pour établir certaines propriétés topologiques de l'ensemble Z (à évaluer son nombre de dimensions, par exemple) sans faire intervenir la notion de rétracte.

1. Soit E_q l'espace euclidien réel à q dimensions. Si $t \in E_1$ et $U=(u_1 \dots u_n) \in E_n$, alors $(t, U) \in E_{n+1}$. Le système des n équations différentielles

$$\frac{du_i}{dt}=f_i(t, u_1 \dots u_n), \quad \text{où} \quad i=1, \dots, n,$$

peut être écrit dans la forme vectorielle

$$(\alpha) \qquad \frac{dU}{dt}=F(t, U),$$

où $F(t, U)=(f_1(t, U), \dots, f_n(t, U))$.

Hypothèse $H(\alpha, \Omega)$. La fonction $F(t, U)$ est réelle et continue dans un ensemble ouvert $\Omega \subset E_{n+1}$. Par chaque point $P=(t^*, U^*)$ passe une intégrale unique $U=V(t, P)$ du système (α). Elle est envisagée dans l'in-

tervalle maximum $\Lambda(P)$ (nécessairement ouvert). Il est défini par les inégalités $\beta(P)<t<\gamma(P)$.

2. Notations et définitions. Posons

$$P=(t^*,U^*)\in\Omega,\quad T(P)=t^*\quad\text{et}\quad W(t,P)=(t,V(t,P)).$$
$$\text{On a } V(t,P)\in E_n \quad\text{et}\quad W(t,P)\in E_{n+1}.$$

Soit $\beta(P)<T(P)-\varepsilon<T(P)<T(P)+\varepsilon<\gamma(P)$. Si t parcourt les intervalles:

$$1^0\ \beta(P)<t<\gamma(P),\quad 2^0\ \beta(P)<t\leqslant T(P),\quad 3^0\ T(P)\leqslant t<\gamma(P),$$
$$4^0\ T(P)-\varepsilon\leqslant t<T(P)\quad\text{et}\quad 5^0\ T(P)<t\leqslant T(P)+\varepsilon,$$

le point $Q=W(t,P)$ décrit les arcs; ils séront désignés respectivement par:

$$1^0\ J(P),\quad 2^0\ J^-(P),\quad 3^0\ J^+(P),\quad 4^0\ J^{-\varepsilon}(P)\quad\text{et}\quad 5^0\ J^{\varepsilon}(P).$$

Les arcs $J(P)$, $J^-(P)$ et $J^+(P)$ constituent respectivement l'intégrale et les demi-intégrales gauche et droite du système (α), issues de P.

Soit ω un ensemble ouvert (appelé tuyau, cf. [1]). Posons $\bar\omega=\omega+B$, où $B=\operatorname{Front}(\omega,\Omega)$ est l'ensemble des points frontières de ω qui appartiennent à Ω.

Soient $P\in B$, $\varepsilon>0$, $K=J^{-\varepsilon}(P)$ et $L=J^{\varepsilon}(P)$. Si, pour un ε suffisamment petit, on a respectivement:

$$1^0\ K\subset\omega,\quad 2^0\ K\subset\omega,\ L\subset\Omega-\bar\omega,\quad 3^0\ L\subset\omega,\quad 4^0\ L\subset\omega,\ K\subset\Omega-\bar\omega$$
$$\text{et}\quad 5^0\ K\subset\Omega-\bar\omega,\ L\subset\Omega-\bar\omega.$$

P sera dit respectivement point:

1^0 de sortie, 2^0 de sortie stricte, 3^0 d'entrée, 4^0 d'entrée stricte, 5^0 de glissement extérieur.

Les classes de ces points seront désignées respectivement par S^*, S, E^*, E et G.

La demi-intégrale droite, pour laquelle $J^+(P)\subset\bar\omega$, sera dite asymptotique (relativement à (α), ω et Ω). L'ensemble Z engendré par toutes les $J^+(P)$ asymptotiques sera dit asymptotique à droite. Lorsque $P\in\omega+E$ et que $J^+(P)$ n'est pas asymptotique, l'on rencontrera en s'avançant sur $J^+(P)$ à partir de P vers la droite l'ensemble B pour la première fois en un point $Q\neq P$ dit conséquent de P et désigné par $C(P)$. En se servant de $J^-(P)$, on définit d'une façon analogue, pour $P\in\omega+S$, l'antécédent $A(P)$. Si $P\in B-E$, on posera $C(P)=P$ et si $P\in B-S$, on posera $A(P)=P$. Toutes ces notions sont relatives au système (α), au tuyau ω et à l'ensemble Ω.

3. Lemme 1. *Considérons quatre ensembles quelconques* $\hat S_1$, $\hat G_1$, $\hat S_2$ *et* $\hat G_2$. *Le point* R_i *parcourant* $\hat S_i+\hat G_i$ *où* $i=1$ *et* 2, *soient* $\varphi_i(R_i)$ *et* $\psi_i(R_i)$ *deux fonctions continues dans* $\hat S_i+\hat G_i$ *et telles que*

$$\varphi_i(R_i) < \psi_i(R_i) \quad pour \quad R_i \in S_i,$$
$$\varphi_i(R_i) = \psi_i(R_i) \quad pour \quad R_i \in \hat{G}_i.$$

Envisageons les ensembles $\Theta_i, \Theta_i^+, \Theta_i^-, \Theta_i^0$ *de points* (σ_i, R_i) *définis par les conditions*:

$$\Theta_i : \varphi_i(R_i) \leqslant \sigma_i \leqslant \psi_i(R_i), \ R_i \in \hat{S}_i + \hat{G}_i;$$
$$\Theta_i^+ : \sigma_i = \psi_i(R_i), \ R_i \in \hat{S}_i;$$
$$\Theta_i^- : \sigma_i = \varphi_i(R_i), \ R_i \in \hat{S}_i;$$
$$\Theta_i^0 : \sigma_i = \varphi_i(R_i) = \psi_i(R_i), \ R_i \in \hat{G}_i$$

et admettons qu'il existe une homéomorphie $R_2 = p(R_1)$ *transformant* $\hat{S}_1 + \hat{G}_1$ *en* $\hat{S}_2 + \hat{G}_2$ *et telle que* $\hat{S}_2 = p(\hat{S}_1)$, $\hat{G}_2 = p(\hat{G}_1)$.

Cela posé, il existe une homéomorphie $(\sigma_2, R_2) = k(\sigma_1, R_1)$ *telle que*

$$\Theta_2 = k(\Theta_1), \ \Theta_2^+ = k(\Theta_1^+), \tag{1}$$
$$\Theta_2^- = k(\Theta_1^-), \ \Theta_2^0 = k(\Theta_1^0).$$

Démonstration. On vérifie facilement que la transformation $(\sigma_2, R_2) = k(\sigma_1, R_1)$ définie pour $R_1 \in \hat{S}_1$ par les formules

$$R_2 = p(R_1),$$
$$\sigma_2 = \varphi_2 p(R_1)) + \frac{\sigma_1 - \varphi_1(R_1)}{\psi_1(R_1) - \varphi_1(R_1)} [\psi_2 p(R_1)) - \varphi_2 p(R_1)]$$

et pour $R_1 \in \hat{G}_1$ par les formules $R_2 = p(R_1)$, $\sigma_2 = \varphi_2(p(R_1))$ est une homéomorphie jouissant des propriétés (1) en question.

4. Hypothèse $K(\alpha, \Omega, \omega)$. La fonction $F(t, U)$ qui figure dans (α) est continue dans l'ensemble ouvert Ω et par chaque point de Ω passe une intégrale unique de (α). Le tuyau ω est ouvert et sa frontière B relative à Ω se compose exclusivement de points d'entrée stricte, de sortie stricte et de glissement extérieur. On a donc (cf. 2) $B = E + S + G$. Les ensembles S et G ne sont pas vides et l'ensemble G est fermé. Chaque point $P \in \overline{\omega} = \omega + B$ admet un antécédent relatif à (α), ω et Ω (cf. 2).

Théorème 1. *Considérons deux systèmes* (α_1) *et* (α_2) *de n équations*

$$(\alpha_1) \qquad \frac{dU_1}{dt_1} = F_1(t_1, U_1),$$

$$(\alpha_2) \qquad \frac{dU_2}{dt_2} = F_2(t_2, U_2).$$

Admettons que (α_i) *satisfait à l'hypothèse* $K(\alpha_i, \Omega_i, \omega_i)$ *où* $i = 1$ *et* 2. *Soit* S_i, E_i, G_i *et* Z_i *respectivement l'ensemble des points de sortie stricte, d'entrée stricte, de glissement extérieur et celui engendré par les demi-intégrales droites asymptotiques relativement à* α_i, ω_i et Ω_i. Posons $B_i = S_i +$

$+E_i+G_i$ et $\Gamma_i=\omega_i-Z_i$. Le point P parcourant S_i+G_i, admettons qu'il existe une homéomorphie $P_2=h(P_1)$ transformant S_1+G_1 en S_2+G_2 de façon que $S_2=h(S_1)$ et $G_2=h(G_1)$.

Cela posé, il existe une homéomorphie $P_2=m(P_1)$ transformant Γ_1 en Γ_2 et telle que

$$S_2=m(S_1),\ G_2=m(G_1),$$
$$E_2\Gamma_2=m(E_1\Gamma_1),\ \omega_2\Gamma_2=m(\omega_1\Gamma_1),\ \Gamma_2=m(\Gamma_1). \tag{2}$$

Démonstration. I. Les fonctions $T(P)$, $W(t,P)$, $A(P)$ et $C(P)$ définies relativement au système (α) passent en $T_i(P_i)$, $W_i(t_i,P_i)$, $A_i(P_i)$ et $C_i(P_i)$ respectivement lorsqu'on remplace le système (α) par le système (α_i).

On vérifie facilement que $P_i\in\Gamma_i$ équivaut à l'existence du conséquent $R_i=C_i(P_i)$ (qui appartient nécessairement à S_i+G_i). Il en résulte facilement que si R_i parcourt l'ensemble S_i+G_i et t_i parcourt l'intervalle $T_i(A_i(R_i))\leqslant t_i\leqslant T_i(R_i)$, l'équation $P_i=W_i(t_i,R_i)$ fournit tous les points $P_i\in\Gamma_i$ et ces points seulement. On a $A_i(R_i)=R_i$ pour $R_i\in S_i+G_i$ lorsque $R_i\in G_i$ et seulement alors.

La fonction $C_i(P_i)$ est continue dans Γ_i (cf. [1]) et, pour les raisons analogues, $A_i(P_i)$ est continue dans ω_i+B_i.

II. Considérons pour un indice i fixe ($i=1$ ou 2) l'ensemble Θ_i des points (t_i,R_i) assujettis aux conditions

$$R_i\in S_i+G_i,\quad T_i(A_i(R_i))\leqslant t_i\leqslant T_i(R_i). \tag{3}$$

Appliquons à ces ensembles le Lemme 1 en posant

$$\sigma_i=t_i,\ \hat{G}_i=G_i,\ \hat{S}_i=S_i,\ \varphi_i(R_i)=T_i(A_i(R_i)),\ \psi(R_i)=T_i(R_i),\ p(R_1)=h(R_1).$$

Les hypothèses de ce Lemme étant alors vérifiées (cf. I), il existe une homéomorphie $(t_2,R_2)=k(t_1,R_1)$, jouissant des propriétés (1). Le point P_i parcourant Γ_i, les formules $t_i=T_i(P_i)$ et $R_i=C_i(P_i)$ fournissent une homéomorphie $(t_i,R_i)=q_i(P_i)$ entre Γ_i et Θ_i (cf. I). On vérifie facilement que la transformation $m=q_2^{-1}kq_1$ est une homéomorphie jouissant des propriétés (2).

5. Voici un théorème analogue au Théorème 1, mais relatif au cas $\omega_1=\omega_2$, $\Omega_1=\Omega_2$.

Théorème 2 *). *Les notations du Théorème 1 étant sauvegardées et le système (α_i) satisfaisant à l'Hypothèse $K(\alpha_i,\Omega,\omega)$, admettons que $S_1=S_2=S$, $G_1=G_2=G$ et $E_1=E_2=E$.*

Ceci posé, il existe une homéomorphie $P_2=m(P_1)$ transformant Γ_1 en Γ_2 et telle que

*) Ce théorème avec quelques conséquences a été communiqué le 23 Octobre 1954 au cours d'un séminaire à Cracovie.

$$P_1 = m(P_1) \quad \text{pour} \quad P_1 \in S + G,$$
$$E\Gamma_2 = m(E\Gamma_1)$$
$$\omega\Gamma_2 = m(\omega\Gamma_1),$$
$$\Gamma_2 = m(\Gamma_1).$$

La démonstration est analogue à celle du Théorème 1. La homéomorphie $h(P_1)$ se réduit maintenant à la transformation par identité.

THÉORÈME 3. *Le système* (α) *satisfaisant à l'Hypothèse* $K(\alpha,\Omega,\omega)$, *soit* Θ *l'ensemble des points* (η,R) *tels que*

$R \in S+G$, $-\varrho(R,G) \leqslant \eta \leqslant 0$, *où* $\varrho(R,G)$ *désigne la distance minimum entre le point* R *et ceux et de l'ensemble* G. *Soit* Γ *l'ensemble des points* $P \in \overline{\omega}$ *qui ne sont situés sur aucune demi-intégrale droite asymptotique relativement à* (α), ω *et* Ω (cf. 2).

Ceci posé, il existe une homéomorphie $P = q(\eta,R)$ *transformant* Θ *en* Γ *et telle que*

$$q(0,R) = R \quad \textit{pour} \quad R \in S+G,$$
$$q(-\varrho(R,G),R) \in E \quad \textit{pour} \quad R \in S,$$
$$q(\eta,R) \subset \Gamma - B \quad \textit{pour} \quad R \in S \quad \textit{et} \quad -\varrho(R,G) < \eta < 0.$$

La démonstration est analogue à celle du Théorème 1. On introduit relativement au système (α) un ensemble Θ^* défini de la même façon que les ensembles (3); puis on applique le Lemme 1 à Θ et Θ^*.

6. Sans admettre l'hypothèse $K(\alpha,\Omega,\omega)$ et sans faire intervenir les ensembles E et G (cf. les définitions et notations du 1 et 2), on peut démontrer le théorème suivant, qui s'applique aussi au cas des systèmes dynamiques.

THÉORÈME 4. *Admettons que le système* (α) *satisfait à l'Hypothèse* $H(\alpha,\Omega)$, *que* ω *est ouvert, que* $\omega \subset \Omega$, *que* $S = S^*$ *(c'est-à-dire que tout point de sortie en est un de sortie stricte) et que* $S_1 \subset S$. *Soient* Γ_1 *la classe des points* $P \in \omega + S_1$ *tels que* $C(P) \in S_1$ *et* Θ *l'ensemble des points* (η,R) *tels que* $-1 < \eta \leqslant 0$ *et* $R \in S_1$.

Cela posé, il existe une homéomorphie $P = \varphi(\eta,R)$ *transformant* Θ *en* Γ_1 *et telle que pour* $R \in S_1$

$$\varphi(0,R) = R \quad \textit{et} \quad \varphi(\eta,R) \in \omega\Gamma_1, \quad \textit{si} \quad -1 < \eta < 0.$$

INSTITUT MATHÉMATIQUE DE L'ACADÉMIE POLONAISE DES SCIENCES

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MATHEMATICS

The analytical characterization of the composed non-homogeneous Poisson process

by

M. FISZ and K. URBANIK

Presented by H. STEINHAUS on January 26, 1955

Let us consider a stochastic process ξ_t defined in the closed time interval $[0, T]$. We shall denote by ξ_I the increment of ξ_t in the interval $I = [a, b]$, where $0 \leqslant a < b \leqslant T$. Let the points $t_0, t_1, \ldots, t_n$, where $a = t_0 < t_1 < \ldots < t_n = b$, divide the interval I in non-overlapping intervals $I_{nk} = [t_{k-1}, t_k)$ for $k = 1, 2, \ldots, n-1$ and $I_{nn} = [t_{n-1}, t_n]$.

We denote

$$\xi_{I_{nk}} = \xi_{t_k} - \xi_{t_{k-1}} \qquad (k = 1, 2, \ldots, n).$$

The following theorem holds:

THEOREM 1. *Let us assume that*:

a) ξ_t *is a process with independent increments*;

β) *there exists for each* $x \neq 0$ *the function* $Q(x,t)$ *defined by the relation*

$$(1) \qquad Q(x,t) = \begin{cases} \lim\limits_{\Delta t \to 0} \dfrac{P(\xi_{t+\Delta t} - \xi_t < x)}{\Delta t} & (x < 0) \\ \lim\limits_{\Delta t \to 0} \dfrac{P(\xi_{t+\Delta t} - \xi_t \geqslant x)}{\Delta t} & (x > 0) \end{cases}$$

and, moreover,

$$(2) \qquad \begin{aligned} &\lim_{x \to -0} Q(x,t) \quad Q_{(-)}(t), \\ &\lim_{x \to +0} Q(x,t) = Q_{(+)}(t), \end{aligned}$$

where the function $Q(x,t)$ is integrable (L) and the convergence in (1) and (2) is uniform with respect to t. Then ξ_t is a composed Poisson process. The logarithm of its characteristic function $\varphi(s, I)$ is given by the formula

$$(3) \qquad \log \varphi(s, I) = \int_{x \neq 0} (e^{isx} - 1)\, d_x \int_I Q(x,t)\, dt.$$

As to the existence of the function $Q(x,t)$ the following theorem is true:

THEOREM 2. *Let us assume that for each $\varepsilon>0$ the relations*

$$\lim_{|I|\to 0} P(|\xi_I|>\varepsilon)=0, \tag{4}$$

$$B(\varepsilon,I)=\lim_{n\to\infty}\sum_{k=1}^{n} P(|\xi_{I_{nk}}>\varepsilon)<\infty \quad \text{for} \quad \max_{1\leqslant k\leqslant n}|I_{nk}|\to 0 \tag{5}$$

hold. Then for each $x\neq 0$ the relation (1) *holds almost everywhere in I.*

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THEORETICAL PHYSICS

Contribution to the final affine field law

by
J. I. HORVÁTH

Presented by L. INFELD on December 7, 1954

Since it became obvious that the geometrical structure of the four-dimensional Riemannian space is determined by the gravitational field alone, many researchers have been studying carefully the possibility of some unified field theory of the first kind *), valid simultaneously for the theory of gravitation and electromagnetism. As these very interesting investigations of the past three decades have shown, practically every generalization of four-dimensional Riemannian geometry may serve as the geometrical basis of such a theory. For this reason numerous theories have been put forward, so that the very recording of the various proposals is a real problem, even without taking into account the theories based on geometries with more than four dimensions. At the present stage of development of field theories every investigation is of great interest, e. g. that published recently by E. Schrödinger [1], or by Kilmister and Stephenson [2], in which some parallelism and identity between different theories is pointed out.

In this paper we shall consider such a possibility in the domain of the affine field theories.

The general idea of our investigation is as follows: In the unified affine field theories a material particle moving solely under the influence of the gravitational and electromagnetic fields has a world line, which can be any of the paths in the space considered. The different field theo-

*) To distinguish between the different kinds of field theories we have proposed the expressions: *field theory of the first and of the second kind*. In the case of a field theory of the second kind the geometrical basis is the four-dimensional Euclidean (or pseudo-Euclidean) space and the physical fields are described by potentials, which are ordinary space-time functions. If, on the other hand, the physical properties of the field are, according to the ideas of Riemann, characterized by the geometrical structure of space, we shall call the theory a field theory of the first kind. From such a point of view Einstein's theory of gravitation e. g. is a field theory of the first kind, and the electromagnetic or mesonic theories are field theories of the second kind.

ries based on affine geometry or space respectively, with different affinity *) but having the same paths, are identical because the world line of the particles is invariant against any changes in the connections which preserve the paths.

1. In a general manifold the parallelism of vectors, defined originally by Levi-Civita for the Riemannian manifold, serves as a basis for the comparison of vectors at different points.

We say that a curve is the focus of points for which the co-ordinates x^i are functions of a parameter t. Let C be any curve and consider the system of differential equations

$$\frac{d\lambda^k}{dt}+\Gamma^i_{j\cdot k}\lambda^j\frac{dx^k}{dt}=0,$$

where $\Gamma^i_{j\cdot k}$ are the affinities and the x's in the Γ's are replaced by the functions of t for C. A set of functions λ^j satisfying this equation are for each value of t the components of a contravariant vector. We say that they are parallel to one another with respect to the curve, and that any one of them may be obtained from any other by a parallel displacement of the latter along the curve. It is well known that the equations given above are not invariant against any change in the affine parameter t. Owing to this, the condition of parallelism is generally given by

$$\lambda^h\left\{\frac{d\lambda^i}{dt}+\Gamma^i_{j\cdot k}\lambda^j\frac{dx^k}{dt}\right\}-\lambda^i\left\{\frac{d\lambda^h}{dt}+\Gamma^h_{j\cdot k}\lambda^j\frac{dx^k}{dt}\right\}=0. \tag{1}$$

The curves whose tangents are parallel with respect to the curves are called the *path of the manifold*. They are an evident generalization of the geodesics of a Riemannian manifold. The general equation of the paths of an affine space is also given by

$$\frac{dx^h}{dt}\left\{\frac{d^2x^i}{dt^2}+\Gamma^i_{j\cdot k}\frac{dx^j}{dt}\frac{dx^k}{dt}\right\}-\frac{dx^i}{dt}\left\{\frac{d^2x^h}{dt^2}+\Gamma^h_{j\cdot k}\frac{dx^j}{dt}\frac{dx^k}{dt}\right\}=0. \tag{1'}$$

It is obvious that this equation is invariant of any change of the affine parameter t and that it is a direct generalization of the one mentioned above.

Let $\Gamma^i_{j\cdot k}$ and $\bar{\Gamma}^i_{j\cdot k}$ be the coefficients of two different connections. Thus it is possible that parallel directions along every curve in the space are the same for the two connections, if, and only if, we take

$$\bar{\Gamma}^i_{j\cdot k}=\Gamma^i_{j\cdot k}+2\,\delta^i_j\psi_k, \tag{2}$$

*) i. e. coefficients of the affine connection.

where ψ_k is an arbitrary vector [3]. Equations (2) define the most general change of connection which preserves parallelism.

2. Hence we can consider the question *whether it is possible to change the affinity preserving parallelism in such a manner that the new components* $\Gamma^{*i}_{j\,k}$ *of the affinity are symmetrical in their indices* j *and* k.

The conditions of the symmetry mentioned are

$$2\Omega^{*i}_{j\cdot k}=2\Omega^{i}_{j\cdot k}+2(\delta_j{}^i\psi_k-\delta_k{}^i\psi_j)=0,$$

where $2\Omega^{*i}_{j\cdot k}$ and $\Omega^{i}_{j\cdot k}$ are the skew-symmetrical parts of the original $\Gamma^{*i}_{j\cdot k}$ and $\Gamma^{i}_{j\cdot k}$, respectively:

$$\Omega^{i}_{j\cdot k}=\tfrac{1}{2}(\Gamma^{i}_{j\cdot k}-\Gamma^{i}_{k\,j}).$$

Unfortunately we have in the four-dimensional case 24 equations for the determination of the four unknowns ψ_k, which generally can only be solved if the original affinities satisfy the relations:

$$\Gamma^{i}_{j\,k}-\Gamma^{i}_{k\cdot j}+\frac{1}{n-1}\{\delta^{i}_{j}(\Gamma^{r}_{k\cdot r}-\Gamma^{r}_{r\cdot k})-\delta_k{}^i(\Gamma^{r}_{j\cdot r}-\Gamma^{r}_{r\cdot j})\}=0. \tag{3}$$

In this case our problem is solved by

$$\psi_j=\frac{1}{2(n-1)}\{\Gamma^{i}_{j\cdot i}-\Gamma^{i}_{i\cdot j}\}, \tag{4}$$

where the space is n-dimensional.

If we have an arbitrarily connected manifold with the affinities $\Gamma^{i}_{j\cdot k}$, we cannot change the affinities so that the new affinities are symmetric in their lower indices, but using the transformation (2) with (4), the introduced star affinity $\Gamma^{*i}_{j\cdot k}$ has a symmetric skew spur:

$$\Gamma^{*i}_{i\cdot l}=\Gamma^{*i}_{i\cdot j}.$$

This change of affinity has the advantage of reducing the number of the unknown components of the affinity and results in a splitting of the Einstein-Ricci tensor (the contracted curvature tensor of the space); inasmuch as by a short computation we get

$$R_{ik}=R^{*}_{ik}+F_{ik},$$

where R_{ik} and R^{*}_{ik} are the Einstein-Ricci tensors of $\Gamma^{i}_{j\cdot k}$ and $\Gamma^{*i}_{i\cdot k}$ respectively, defined by

$$R_{ik}=-\partial_r\Gamma^{r}_{i\cdot k}+\partial_k\Gamma^{r}_{i\cdot r}+\Gamma^{r}_{i\cdot s}\Gamma^{s}_{r\cdot k}-\Gamma^{s}_{r\cdot s}\Gamma^{r}_{i\cdot k}$$

and

$$F_{ik} = \frac{z}{n-1}(\partial_i \psi_k - \partial_k \psi_i).$$

Thus we find that our affinities Γ^{*i}_{jk} are identical with the star affinities introduced by Schrödinger as an *ad hoc* abbreviation [4].

3. The above considerations are based on the geodetic axiom of the theory of relativity. But the modern theory of motion does not use this axiom. In the case of the general theory of relativity it is possible, on the basis of the investigations of Einstein, Infeld and Hoffmann [5], to reduce the equations of motion from the field equations.

Unfortunately, we have not yet found the final unified theory of fields and it is also not sure that the final unified theory will be based on the affine space. The final field equations have not yet been established either, and this has consequently hindered investigations into the deduction of equations of motion from field equations, i. e. such investigations have not yielded satisfactory results [6].

In this paper we should like to point out that the equations of motion in their parameter-invariant form must be invariant against changes of the affinities which preserve the parallelism. Unfortunately, however, it is obvious that the field equations of any possible different affine theory of fields have not *generally* this property of invariance and it is questionable whether such an affine theory, the field equations of which in their parameter-invariant form are invariant against any change of the affinities, can exist at all.

However, it seems to be obvious that a deduction of the equations of motion from the field equations in the case of the final theory must exist, in consequence of which there is a strong correspondence between the field equations and the equations of motion. Now, if the basic idea of this paper can be accepted, *the equations of motion in their parameter invariant form must be invariant against changes of the affinities which preserve the parallelism* and as a consequence of this theorem, based on the above considerations, *we have also to postulate the invariance of the final field equations in their parameter-invariant form against such transformations.*

This postulate seems to be a selection principle for the final theory in the case of the affine theories.

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DEPARTMENT OF THEORETICAL PHYSICS OF THE UNIVERSITY OF SEGED.

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The diffraction of a dipole field by a perfectly conducting wedge

by

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A method is given for solving the diffraction problem of an arbitrary electric or magnetic dipole by a perfectly conducting wedge of arbitrary opening angle. The method consists in a generalization and modification of the method due to Senior [5].

The calculations are made for the H_ν field of a magnetic dipole Π_y^m for an outer angle of the wedge $\varkappa \geqslant \pi$.

I. In this paper a method is given for solving the diffraction problem of the dipole field by a conducting wedge. The method is a generalization and modification of the method devised by Senior [5] for a conducting half-plane. The modification was dictated by the necessity of avoiding certain mathematical steps which are not allowed. The essential feature of the method consists in a specific transition from Sommerfeld's solution [1], [2] for plane waves diffraction to the dipole field diffraction. The generality of the method rests on: (i) the arbitrariness of orientation of the (electric or magnetic) dipole with respect to the wedge, (ii) the arbitrariness of the wedge angle. Bearing this in mind, the only solution given here is that for the $H_y(r,\theta,z)$ component of a magnetic Π_y^m dipole in the point (r_0,θ_0,z_0). The outer angle of the wedge $\varkappa$ is assumed to be greater than π. The edge of the wedge is coincident with the z axis (Fig. 1).

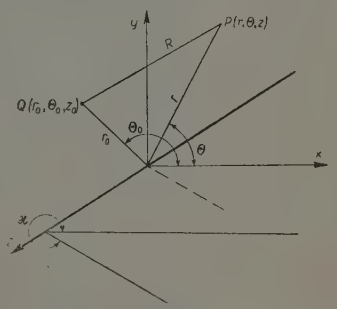

Fig. 1.

II. The Hertz vector of a dipole in free space can be represented by the following development in plane waves:

$$(1)\qquad \frac{e^{-ikR}}{kR} = \frac{1}{2\pi i}\int_{-i\infty}^{\pi/2} \cos\beta\, d\beta \int_{-\pi}^{\pi} d\alpha\, e^{ik[r\cos(\theta-\alpha)\cos\beta - r_0\cos(\theta_0-\alpha)\cos\beta - (z-z_0)\sin\beta]}$$

where: $R = \sqrt{r^2+r_0^2+(z-z_0)^2-2rr_0\cos(\theta-\theta_0)}$; $z-z_0 \geqslant 0$.

The integral on the path $(-\pi,\pi)$ for α can be changed to the sum of the two other integrals on paths $(-i\infty,\pi+i\infty)$ and $(-\pi+i\infty,-i\infty)$. Accordingly we write (1) as the sum of the two terms K' and I':

$$(2)\qquad \frac{e^{-ikR}}{kR}=K'+S_{I'}I'^{x}\underset{\text{def}}{=}\left[K\left(\theta_0;\ \alpha=P\left(\frac{\pi}{2}\right)\right)+\right.$$
$$\left.+S_{I'}I^{x}\left(\theta_0;\ \alpha=P\left(\frac{\pi}{2}\right)\right)\right]e^{ik[r\cos(\theta-\alpha)\cos\beta-z\sin\beta]},$$

where: I'^{x} is a conjugate complex of I', and $S_{I'}$ denotes the operator of conjugation acting only on I'; operators K and I have the following forms:

$$K\left(\theta_0;\ \alpha=P\left(\frac{\pi}{2}\right)\right)=\int_{-i\infty}^{\pi/2}\cos\beta\,d\beta\int_{P(\pi/2)}da\,F(\alpha\beta),$$

$$I^{x}\left(\theta_0;\ \alpha=P\left(\frac{\pi}{2}\right)\right)=\int_{-i\infty}^{-\pi/2}\cos\beta\,d\beta\int_{P(\pi/2)}da\,F(\alpha\beta),$$

$$\left(F(\alpha\beta)=e^{-ik[r_0\cos(\theta_0-\alpha)-z_0\sin\beta]}\right).$$

In the integrals K and I^{x} the variable α traverses the path from $-i\infty$ to $\pi+i\infty$, which can be, in particular, the path of the steepest descent going through the point $\alpha=\frac{\pi}{2}$; we denote this path of steepest descent by $P\left(\frac{\pi}{2}\right)$. The meaning of the representation $\frac{e^{-ikR}}{kR}$ with the help of the quantity I'^{x} conjugated to I' will be explained below.

The field H_y of a magnetic dipole Π_y^m will be expressed according to (2) by

$$(3)\qquad H_y=\left(\frac{\partial^2}{\partial y\partial y_0}-k^2\right)K'+\left(\frac{\partial^2}{\partial y\partial y_0}-k^2\right)S_{I'}I'^{x}.$$

III. The idea of the Senior method is to start from the plane wave solution. Such a solution can be generalized for the three dimensions by replacing k by $k\cos\beta$ and multiplying by $e^{-ikz\sin\beta}$. The solutions obtained in this way can be considered as a z component of the electric (V_z^e) or magnetic (V_z^m) Hertz vector.

In this way we get from the plane waves

$$(4)\qquad E_z^0=e^{ikr\cos(\theta-\alpha)}$$

a Hertz vector of the electric type V_z^e:

$$(5)\qquad V_z^e=e^{ik[r\cos(\theta-\alpha)\cos\beta-z\sin\beta]}$$

and from H_z^0 we get in the same way the Hertz vector V_z^m.

The electromagnetic field derived from V_z^e shall be denoted by $\vec{E}^e$, $\vec{H}^e$; similarly, the field from V_z^m by $\vec{E}^m$, $\vec{H}^m$. Both fields can be combined with arbitrary constants λ and μ: $\lambda\vec{E}^e+\mu\vec{E}^m$, $\lambda\vec{H}^e+\mu\vec{H}^m$.

Since we are limiting ourselves to an examination of the field of the magnetic dipole Π_y^m, we remark at the outset that the E_y component of this field vanishes in free space. The identical vanishing of E_y is secured by the following choice of constants λ and μ: $\lambda=\cos\alpha$, $\mu=-\sin\alpha\sin\beta$. In this way we get the superposition of the form

$$\vec{E}=\cos\alpha\vec{E}^e-\sin\alpha\sin\beta\vec{E}^m,\ \vec{H}=\cos\alpha\vec{H}^e-\sin\alpha\sin\beta\vec{H}^m. \tag{6}$$

Now, by appropriate integration of (6) over the angles α and β (this can be done by K and I^x operators) we can get the electromagnetic field of the dipole Π_y^m. For instance, w ehave for the H_y component (given by (3)),

$$k^2[K+S_{I'}I^x][\cos\alpha H_y^e-\sin\alpha\sin\beta H_y^m]=H_y, \tag{7}$$

where:

$$H_y^e=\frac{i}{k}\frac{\partial}{\partial x}V_z^e,\quad H_y^m=\frac{-i}{k}\sin\beta\frac{\partial}{\partial y}V_z^m. \tag{8}$$

V_z^e and V_z^m are expressed by formula (5).

The above method of obtaining the vacuum field of the dipole can be transferred to obtain the diffraction field on the wedge. For this purpose one has to go out from the diffraction of plane waves E_z and H_z on the wedge, and hence, by generalization to three dimensions, one can get V_z^e and V_z^m. Such a transition is permissible owing to the fact that boundary conditions on the perfect conductor are the same for E_z and V_z^e as for H_z and V_z^m.

IV. Starting from Sommerfeld's solution of wedge diffraction [2], [3] we can represent E_z and H_z as follows:

$$E_z=e^{ikr\cos(\theta-\alpha)}-\delta(2\varkappa-2\pi\leqslant\alpha\leqslant2\varkappa)e^{ikr\cos(2\varkappa-\theta-\alpha)}+$$

$$+\frac{i}{2\varkappa}\int_C d\gamma\, e^{-ikr\cos(\theta\mp\gamma)}\left[\frac{\sin\frac{\pi}{\varkappa}\alpha}{\cos\frac{\pi}{\varkappa}(\pi-\gamma)-\cos\frac{\pi}{\varkappa}\alpha}-\frac{\sin\frac{\pi}{\varkappa}\alpha}{\cos\frac{\pi}{\varkappa}(\gamma_0-\gamma)-\cos\frac{\pi}{\varkappa}\alpha}\right], \tag{9}$$

$$H_z=e^{ikr\cos(\theta-\alpha)}+\delta(2\varkappa-2\pi\leqslant\alpha\leqslant2\varkappa)e^{ikr\cos(2\varkappa-\theta-\alpha)}+$$

$$\mp\frac{i}{2\varkappa}\int_C d\gamma\, e^{-ikr\cos(\theta\mp\gamma)}\left[\frac{\sin\frac{\pi}{\varkappa}(\pi-\gamma)}{\cos\frac{\pi}{\varkappa}(\pi-\gamma)-\cos\frac{\pi}{\varkappa}\alpha}-\frac{\sin\frac{\pi}{\varkappa}(\gamma_0-\gamma)}{\cos\frac{\pi}{\varkappa}(\gamma_0-\gamma)-\cos\frac{\pi}{\varkappa}\alpha}\right]. \tag{10}$$

$$\gamma_0=-\pi\text{ for }\theta<\pi,\ \gamma_0=3\pi\text{ for }\theta>\pi.$$

Here the upper signs refer to $\theta<\pi$, the lower to $\theta>\pi$; $\delta(\ )$ equals unity when the inequality given in the bracket is fulfilled, and equals zero in the opposite case. The contour C is shown on Fig. 2. Formulas (9) and (10) are a generalization of the Clemmov representation for half-plane diffraction of plane waves [4].

V. The component H_y of the diffraction field of the dipole Π_y^m can now be obtained from (7) and (8). However, the V_z^e and V_z^m in (8) are now to be deduced from the wedge diffraction fields E_z and H_z, eqs. (9), (10), by the method outlined in III.

The H_y obtained in this way will be expressed, after some transformations, in the following form

$$H_y=\left(\frac{\hat{c}^2}{\hat{c}y\,\hat{c}y_0}-k^2\right)\frac{e^{-ikR}}{kR}+\delta\left(\varkappa\geqslant\frac{5}{4}\pi\right)\left(\frac{\partial^2}{\hat{c}y\hat{c}y_0}+k^2\cos 2\varkappa\right)\frac{e^{-ikS^1}}{kS^1}+$$

$$+\left[K\left(\theta_0;\ \alpha=P\left(\frac{\pi}{2}\right)\right)+S_{I'}I^x\left(\theta_0;\ \alpha=P\left(\frac{\pi}{2}\right)\right)\right]\frac{i}{2\varkappa}\int_C d\gamma\, e^{ik[r\cos(\theta\mp\gamma)\cos\beta-z\sin\beta]}\cdot$$

$$\cdot\left\{k^2\left[\frac{\cos\gamma\cos\alpha\sin\frac{\pi}{\varkappa}\alpha-\sin\gamma\sin\alpha\sin\frac{\pi}{\varkappa}(\pi-\gamma)}{\cos\frac{\pi}{\varkappa}(\pi-\gamma)-\cos\frac{\pi}{\varkappa}\alpha}-\right.\right.$$

(11)

$$\left.\left.-\frac{\cos\gamma\cos\alpha\sin\frac{\pi}{\varkappa}\alpha-\sin\gamma\sin\alpha\sin\frac{\pi}{\varkappa}(\gamma_0-\gamma)}{\cos\frac{\pi}{\varkappa}(\gamma_0-\gamma)-\cos\frac{\pi}{\varkappa}\alpha}\right]\mp\right.$$

$$\left.\mp\frac{\hat{c}^2}{\hat{c}y\hat{c}y_0}\left[\frac{\sin\frac{\pi}{\varkappa}(\pi-\gamma)}{\cos\frac{\pi}{\varkappa}(\pi-\gamma)-\cos\frac{\pi}{\varkappa}\alpha}-\frac{\sin\frac{\pi}{\varkappa}(\gamma_0-\gamma)}{\cos\frac{\pi}{\varkappa}(\gamma_0-\gamma)-\cos\frac{\pi}{\varkappa}\alpha}\right]\right\},$$

where: $S^1=\sqrt{r^2+r_0^2+(z-z_0)^2-2rr_0\cos(\theta+\theta_0-2\varkappa)}$; the operator $\frac{\partial^2}{\hat{c}y\partial y_0}$ acts to the right as well as to the left; the upper sign is for $\theta<\pi$, the lower for $\theta>\pi$. By means of a procedure analogous to that given by Senior [5], we deform the path C for the γ variable (Fig. 2) to the steepest descent path going through the saddle point $\gamma=\theta$ for $\theta<\pi$ and $\gamma=2\pi-\theta$ for $\theta>\pi$. Likewise, we shift the path $P\left(\frac{\pi}{2}\right)$, traversed by α, to the saddle point $\alpha=\theta_0$ ($\theta_0<\pi$). Both operations are allowed on account of the vanishing of the integrand in the appropriate real interval for $\mp i\infty$. This holds for K as well as for I^x. Notice that the term $\cos\beta$ in the exponent in K' and I'^x is always greater than zero. In the original Senior method

for a halfplane the distortion of paths is not allowable since $\cos\beta$ changes sign (β runs there from $-i\infty$ to $\pi+i\infty$). Here we see the meaning of the use of the conjugate complex quantity I'^{x}. The essential of the method consists therefore in performing the operation deforming the paths of α and γ in the integral I'^{x} and not in I'.

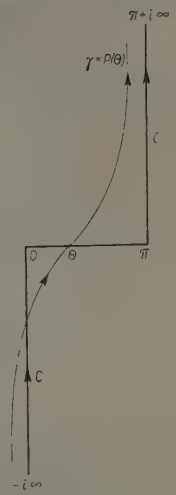

Fig. 2

The poles which appear by the shifting of the integration paths give, together with two first terms of (11), the so-called geometrical optic field H_y^g.

The expression given by the integrals similar to the ones appearing in (11), but with other contours for α and γ, gives the so-called pure diffraction field H_y^d.

The field H_y^g is given by (for $\varkappa \geqslant \pi$):

$$H_y^g = \delta(0 \leqslant \theta \leqslant \pi+\theta_0)\left(\frac{\partial^2}{\partial y \partial y_0} - k^2\right)\frac{e^{-ikR}}{kR} +$$

$$+\delta(0 \leqslant \theta \leqslant \pi-\theta_0)\left(\frac{\partial^2}{\partial y \partial y_0} + k^2\right)\frac{e^{-ikS^0}}{kS^0} +$$

$$+\delta\left(\pi \leqslant \varkappa \leqslant \frac{\pi+\theta_0+\theta}{2}\right)\left(\frac{\partial^2}{\partial y \partial y_0} + k^2\cos 2\varkappa\right)\frac{e^{-ikS^1}}{kS^1} \tag{12}$$

where: S^k are the distances from the point (r,θ,z) to (r_0,θ_0^k,z_0) and θ_0^k equal: $-\theta_0, 2\varkappa-\theta_0$ respectively.

In the following we deal with the diffraction field H_y^d. After performing some simple transformations and going from variables α,β,γ to α,β,ϱ ($\varrho=\alpha+\gamma$) one can see that

$$H_y^d = [K(0;\alpha=P(0)) + S_{l'} I^x(0;\alpha=P(0))]\frac{i}{4\varkappa}\int_{P(0)} d\varrho\, e^{ik[r\cos(\varrho-\alpha)\cos\beta - z\sin\beta]}\cdot$$

$$\cdot \sum_{l=1,2}\left(k^2(-)^l\cos(\varrho+\varphi_l-\pi) - \frac{\partial^2}{\partial y\partial y_0}\right)\left(\operatorname{ctg}\frac{\pi}{2\varkappa}(\varrho+\varphi_l-\pi) - \operatorname{ctg}\frac{\pi}{2\varkappa}(\varrho+\varphi_l-\gamma_0)\right), \tag{13}$$

where the ϱ variable runs through the steepest descent path $P(0)$ [5]; the operator $\partial^2/\partial y\partial y_0$ acts to the right and to the left; $\varphi_1=\theta+\theta_0$; $\varphi_2=\theta-\theta_0$, $\gamma_0=-\pi$ for $\theta<\pi$, $\gamma_0=3\pi$ for $\theta>\pi$.

Using (2) we get for the field H_y^d:

$$H_y^d = \frac{i}{4\varkappa}\int_{P(0)} d\varrho \sum_{l=1,2}\left(k^2(-)^l\cos(\varrho+\varphi_l-\pi) - \frac{\partial^2}{\partial y\partial y_0}\right)\cdot$$

$$\cdot \frac{e^{-ikR_\varrho}}{kR_\varrho}\left(\operatorname{ctg}\frac{\pi}{2\varkappa}(\varrho+\varphi_l-\pi) - \operatorname{ctg}\frac{\pi}{2\varkappa}(\varrho+\varphi_l-\gamma_0)\right), \tag{14}$$

where: $R_\varrho = \sqrt{r^2+r_0^2+(z-z_0)^2+2rr_0\cos\varrho}$.

The total field is given by the sum: $H_y^g + H_y^d$ (see eqs. (12) and (14)). A full account of this work, together with approximate formulas for short wave lengths, will be given in Acta Physica Polonica.

I wish to express my thanks to Professor L. Infeld and to Professor W. Rubinowicz for their valuable remarks and discussions.

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Die Bewegungsgleichungen in der allgemeinen Relativitätstheorie und die Koordinatenbedingung

von

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Vorgelegt von L. INFELD am 26. Januar 1955

1. Das Problem der Bewegungsgleichungen in der allgemeinen Relativitätstheorie kann man für den sogenannten astronomischen Fall — d. h. für ein System von Körpern, welche ein schwaches Gravitationsfeld erzeugen und sich mit kleinen Geschwindigkeiten bewegen — als gelöst betrachten. Für die Ableitung der Bewegungsgleichungen wurden drei verschiedene Methoden entwickelt: die Methode von Einstein, Infeld und Hoffman [1], die von Fock [2] und Petrova [3] und die von Papapetrou [4]. Aus allen diesen Methoden ergeben sich dieselben Bewegungsgleichungen, sowohl in der Newtonschen (ersten), als auch in der darauffolgenden (zweiten) Näherung.

Es besteht jedoch immer noch eine Meinungsverschiedenheit über die Rolle, welche die bei diesen Rechnungen verwendete Koordinatenbedingung spielt. Fock hält eine Koordinatenbedingung zur eindeutigen Bestimmung der Bewegungsgleichungen schon in der Newtonschen Näherung für notwendig. Nach Infeld [5] hingegen sind die Bewegungsgleichungen sowohl in der ersten als auch in der nächsten Näherung allein durch die Näherungsmethode eindeutig bestimmt. Keinen dieser beiden Standpunkte teilen wir ganz. Wie der eine von uns in einer früheren Arbeit [6] gezeigt hat, ist in der Newtonschen Näherung eine Koordinatenbedingung überflüssig, nicht aber in den höheren Näherungen.

Kürzlich hat Infeld [7] erneut die Rolle der Koordinatenbedingung diskutiert und bewiesen, dass die Bewegungsgleichungen in der Newtonschen Näherung von der Koordinatenbedingung unabhängig sind. Dabei macht er neben anderen von den folgenden zwei Annahmen Gebrauch, die für die Methode von Einstein, Infeld und Hoffman charakteristisch sind:

(a) Die Potenzreihen, in welche sich die Grössen $g_{\mu\nu}$ und $\gamma_{\mu\nu}$ *) entwickeln lassen, enthalten nur gerade, bzw. nur ungerade Potenzen, je nachdem beide oder keiner, bzw. nur einer, der Indizes μ, ν gleich Null ist.

(b) Die Entwicklung von γ_{00} beginnt mit dem Term zweiter Ordnung, diejenige von γ_{0i} mit dem Term dritter und die von γ_{ik} mit dem Term vierter Ordnung **).

Wie schon in der Arbeit [6], allerdings in allzu gedrängter Form, bewiesen wurde, folgt aus der Annahme (b) nicht nur die Eindeutigkeit der Bewegungsgleichungen in der Newtonschen, sondern auch die in der zweiten Näherung. Wir kommen auf diesen Punkt in einer nächsten Mitteilung zurück. Hier wollen wir den Beweis dafür erbringen, dass sich die Bewegungsgleichungen erster Näherung auch ohne die beiden Annahmen (a) und (b) in unveränderter Form ergeben und von einer eventuell verwendeten Koordinatenbedingung unabhängig sind.

2. Der Kern dieses Beweises ist in der schon erwähnten Arbeit [6] enthalten. Es wurde dort die Bewegungsgleichung eines mechanischen Kontinuums aus den Feldgleichungen der allgemeinen Relativitätstheorie und der dynamischen Gleichung abgeleitet. Hier wollen wir zunächst die jener Rechnung zugrundegelegten drei Annahmen zusammenfassen und ihre physikalische Bedeutung erörtern.

(I) Das Gravitationsfeld sei schwach. Der metrische Fundamentaltensor soll sich also in eine Potenzreihe nach einem kleinen Parameter entwickeln lassen. Der dem vorliegenden Problem angepasste Parameter ergibt sich aus der nachfolgenden Annahme (II) und ist von der Grössenordnung des Verhältnisses v/c, wobei v der Betrag der Geschwindigkeit der Massenelemente ist. Formal dürfen wir als Parameter die Grösse l/c verwenden. Das Anfangsglied der Entwicklung ist, wegen der geforderten euklidischen Verhältnisse im Unendlichen, der metrische Tensor der speziellen Relativitätstheorie $\eta_{\mu\nu}$ ($\eta_{ik} = -\delta_{ik}$, $\eta_{0i} = 0$, $\eta_{00} = 1$). Es soll also gelten

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{c}\,{}_{1}g_{\mu\nu} + \frac{1}{c^2}\,{}_{2}g_{\mu\nu} + \dots$$

(II) Die Geschwindigkeit $v^i = dx^i/dt$ der Massenelemente sei klein gegenüber der Lichtgeschwindigkeit,

$$|v| \ll c.$$

Das Gravitationsfeld wird dann quasi-statisch sein. Die Ableitung irgendeiner Feldgrösse nach der Zeitkoordinate x^0 wird also um eine

*) Für die Definition von $\gamma_{\mu\nu}$ vgl. z. B. Infeld [7].

**) Wir schliessen uns der üblichen Verabredung an, wonach lateinische Indizes nur die Werte 1, 2, 3, dagegen griechische Indizes die Werte 1, 2, 3 und 0 durchlaufen.

Grössenordnung kleiner sein als die Ableitungen nach dem Raumkoordinaten x^i *),

$$g_{\mu\nu,0} \sim \frac{1}{c} g_{\mu\nu,i}$$

Es sei noch bemerkt, dass die Annahme (II) nur in Verbindung mit (I) aufrechterhalten werden kann. Die Bedingung $|v| \ll c$ kann nämlich nur dann dauernd erfüllt sein, wenn auch die Beschleunigung klein ist. Und dies wird durch die Annahme (I) gewährleistet.

(III) Die das Gravitationsfeld erzeugende Materie sei ein allgemeines, mechanisches Kontinuum. Das Anfangsglied der Entwicklung des mechanischen Tensors habe dieselbe Gestalt, wie in der speziellen Relativitätstheorie für $|v| \ll c$. Es sei also

$$T^{00} = \varrho c^2 + \ldots, \quad T^{0i} = \varrho c v^i + \ldots, \quad T^{ik} = \varrho v^i v^k + p^{ik} + \ldots,$$

wobei ϱ die Materiedichte und p^{ik} einen (3-dimensionalen) Spannungstensor bedeuten.

Diese letzte Voraussetzung ist in den Rechnungen von Fock-Petrova und Papapetrou explizite enthalten. In der Methode von Einstein, Infeld und Hoffman wird sie nicht unmittelbar benutzt. Sie war aber auch dort erforderlich, um die Einführung der für diese Methode charakteristischen Annahme (b) zu rechtfertigen.

3. Die Newtonsche Bewegungsgleichung des Kontinuums folgt direkt aus den Feldgleichungen der allgemeinen Relativitätstheorie und der dynamischen Gleichung, wenn man noch die Annahmen (I), (II) und (III) berücksichtigt, ohne dass irgendeine Koordinatenbedingung erforderlich ist. Die dazu nötige Rechnung wurde in der Arbeit [6] durchgeführt. Hier sei nur der allgemeine Gang dieser Rechnung beschrieben.

Die Feldgleichungen lauten

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T\right), \tag{1}$$

wobei G die Newtonsche Gravitationskonstante ist. Beachtet man die Annahme (I), so ergibt sich auch für $R_{\mu\nu}$ eine ähnliche Entwicklung:

$$R_{\mu\nu} = \frac{1}{c} \underset{1}{R}_{\mu\nu} + \frac{1}{c^2} \underset{2}{R}_{\mu\nu} + \ldots$$

Dagegen ist die rechte Seite von (1) wegen (III) mindestens von der Ordnung $1/c^2$. Es muss daher gelten

$$\underset{1}{R}_{\mu\nu} = 0.$$

*) Als Zeitkoordinate benutzen wir $x^0 = ct$; ferner sei $(\,)_{,\mu} = \frac{\partial}{\partial x^\mu}(\,)$.

Die allgemeine Lösung dieser Gleichung lautet

$$(2) \qquad \underset{1}{g_{00}} = 0, \quad \underset{1}{g_{0i}} = a_{0,i}, \quad \underset{1}{g_{ik}} = a_{i,k} + a_{k,i},$$

wobei die a_μ vier beliebige Funktionen von x^μ sind.

Weiter finden wir aus (I) und (2)

$$\underset{2}{R_{00}} = -\Delta(\tfrac{1}{2}\underset{2}{g_{00}} - a_{0,t}).$$

Andererseits folgt aus (1)

$$\underset{2}{R_{00}} = 4\pi G\varrho.$$

Setzen wir also

$$(3) \qquad u \equiv -\tfrac{1}{2}\underset{2}{g_{00}} + a_{0,t},$$

so erhalten wir für u die Poisson-Gleichung

$$(4) \qquad \Delta u = -4\pi G\varrho.$$

D. h. u ist mit dem Newtonschen Gravitationspotential der Massenverteilung ϱ identisch.

Die schon berechneten Grössen $\underset{1}{g_{\mu\nu}}$ und $\underset{2}{g_{00}}$ reichen für die Bestimmung der Bewegungsgleichung in erster Näherung aus. Es genügt dafür, dass wir die dynamische Gleichung,

$$(5) \qquad \mathfrak{T}^{\nu}_{\mu,\nu} - \tfrac{1}{2}\mathfrak{T}^{\alpha\beta}g_{\alpha\beta,\mu} = 0,$$

berücksichtigen. Für $\mu = 0$ ergibt sich daraus in der ersten Näherung, unter Beachtung von (III) und (2), der Erhaltungssatz der Masse

$$(6) \qquad \frac{d}{dt}(\varrho\delta V) = 0.$$

Für $\mu = i$ liefert uns dann Gleichung (5) die eigentliche Bewegungsgleichung:

$$\varrho\frac{dv^i}{dt} + p^{ik}{}_{,k} = \varrho(-\tfrac{1}{2}\underset{2}{g_{00}} + a_{0,t})_{,i}.$$

Diese lässt sich mit (3) auch schreiben

$$(7) \qquad \varrho\frac{dv^i}{dt} + p^{ik}{}_{,k} = \varrho u_{,i}.$$

Das ist aber gerade die Bewegungsgleichung der Newtonschen Theorie für das Massenelement eines allgemeinen Kontinuums, das unter der Einwirkung der Gravitationskraft $\varrho u_{,i}$ und der Spannungen p^{ik} steht. Es sei nochmals betont, dass dieses Ergebnis aus den Feldgleichungen und der dynamischen Gleichung mit Hilfe der Annahmen (I), (II) und (III) abgeleitet wurde. Eine Koordinatenbedingung war dabei nicht erforderlich.

4. Es bleibt noch zu zeigen, dass die Bewegungsgleichung (7) des Kontinuums ohne weiteres zu den von Einstein, Infeld und Hoffman sowie von Fock abgeleiteten Bewegungsgleichungen erster Näherung führt. Dies geschieht mit Hilfe der folgenden Rechnung, die völlig im Rahmen der Newtonschen Gravitationstheorie verläuft.

Wir gehen zu dem Fall eines Systems voneinander getrennter Körper über, indem wir annehmen, dass der Materietensor $T^{\mu\nu}$ also in erster Näherung die Massendichte ϱ und die Spannungen p^{ik} nur innerhalb der voneinander getrennten (3-dimensionalen) Gebiete $V_1, V_2, \ldots, V_n$ von Null verschieden ist. Darüber hinaus soll das Volumen V_1 kugelförmig und die Dichte ϱ in ihm kugelsymmetrisch sein. Zur Ableitung der Bewegungsgleichung dieses Körpers genügt es, Gleichung (7) über das Volumen V_1 zu integrieren.

Es gilt zunächst

$$M_1 R_1^i(t) = \int\limits_{V_1(t)} x_1^i \varrho(\mathfrak{r}_1, t)\, dV_1,$$

wo $M_1 = \int\limits_{V_1(t)} \varrho dV_1$ die Gesamtmasse und R_1^i die Koordinaten des Schwerpunktes des ersten Körpers sind. Damit können wir den ersten Term der linken Seite von (7) unter Beachtung von (6) schreiben

$$\int\limits_{V_1} \varrho(\mathfrak{r}_1, t) \frac{dv^i}{dt} dV_1 = M_1 \frac{d^2 R_1^i(t)}{dt^2}.$$

Der zweite Term links verschwindet, denn er lässt sich als divergenzartig mit Hilfe des Gausschen Satzes in ein Oberflächenintegral umwandeln.

Die rechte Seite von (7) verlangt eine ausführlichere Rechnung. Nach (4) hat u die Gestalt

$$u(\mathfrak{r}, t) = \sum_{\lambda=1}^{n}{}' G \int\limits_{V_\lambda} \frac{\varrho(\mathfrak{r}_\lambda, t)}{|\mathfrak{r} - \mathfrak{r}_\lambda|} dV_\lambda \equiv \sum_{\lambda=1} u_\lambda(\mathfrak{r}, t). \tag{8}$$

Damit zerlegt sich auch die rechte Seite von (7) in n Summanden.

Das Integral über den Term mit $\lambda = 1$ beschreibt die gesamte Gravitationskraft der Massenverteilung in V_1 auf sich selbst und muss daher verschwinden. Somit haben wir die über V_1 integrierte Gleichung zunächst auf folgende From gebracht:

$$\begin{aligned} M_1 \frac{d^2 R_1^i(t)}{dt^2} &= \sum_{\lambda=2}^{n}{}' G \int\limits_{V_1} \int\limits_{V_\lambda} \varrho(\mathfrak{r}_1, t)\, \varrho(\mathfrak{r}_\lambda, t) \frac{\partial}{\partial x_1^i} \frac{1}{|\mathfrak{r}_1 - \mathfrak{r}_\lambda|} dV_1 dV_\lambda = \\ &= -\sum_{\lambda=2}^{n}{}' G \int\limits_{V_\lambda} \varrho(\mathfrak{r}_\lambda, t) \frac{\partial}{\partial x_\lambda^i} \left\{ \int\limits_{V_\lambda} \frac{\varrho(\mathfrak{r}_1, t)}{|\mathfrak{r}_1 - \mathfrak{r}_\lambda|} dV_1 \right\} dV_\lambda. \end{aligned} \tag{9}$$

Wegen der Kugelsymmetrie der Massenverteilung in V_1 gilt nun folgende Beziehung:

$$\int_{V_1} \frac{\varrho(\mathfrak{r}_1,t)}{|\mathfrak{r}_1-\mathfrak{r}_\lambda|}\,dV_1 = M_1 \frac{1}{|\mathfrak{R}_1-\mathfrak{w}_\lambda|}. \tag{10}$$

Die rechte Seite von (9) lautet also

$$-M_1 \sum_{\lambda=2}^{n}{}' G \int_{V_\lambda} \varrho(\mathfrak{r}_\lambda,t)\,\frac{\partial}{\partial x_\lambda^i}\,\frac{1}{|\mathfrak{R}_1-\mathfrak{r}_\lambda|}\,dV_1 = M_1 \sum_{\lambda=2}^{n}{}' G\,\frac{\partial}{\partial R_1^i}\int_{V_\lambda}\frac{\varrho(\mathfrak{r}_\lambda,t)}{|\mathfrak{R}_1-\mathfrak{r}_\lambda|}\,dV_\lambda =$$

$$= M_1 \frac{\partial}{\partial R_1^i} \sum_{\lambda=2}^{n}{}' u_\lambda(\mathfrak{R}_1,t).$$

Setzen wir diesen Wert in (9) ein, so ergibt sich die Bewegungsgleichung des ersten Körpers zu

$$\frac{d^2R_1^i(t)}{dt^2} = \frac{\partial}{\partial R_1^i} \sum_{\lambda=2}^{n}{}' u_\lambda(\mathfrak{R}_1,t). \tag{11}$$

Dies stimmt mit der von Einstein, Infeld und Hoffman sowie mit der von Fock abgeleiteten Bewegungsgleichung genau überein.

Um zu dem ganzen System der von diesen Autoren errechneten Bewegungsgleichungen zu gelangen, genügt es offenbar anzunehmen, dass die Massenverteilung in jedem der n betrachteten Körper kugelsymmetrisch ist.

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EXPERIMENTAL PHYSICS

The influence of ultra-violet irradiation on the absorption in the infra-red region of KBr and KI crystals with anionic impurities

by
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Communicated by A. SOŁTAN at the meeting of December 13, 1954

The optical properties of ionic crystals are to a considerable degree dependent on the impurities, which are present in the crystal lattice. It is already well-known, that the foreign cations embedded in the crystal lattice of the base substance give rise to absorption bands, and that the absorption in these bands generally produces luminescence. The properties of ionic crystals activated with foreign cations have been the subject of detailed investigation for some twenty years or so. But as regards the optical properties of ionic crystals in which anionic impurities are present, there are less abundant data available.

Alkali halide crystals with anionic impurities are characterized by absorption bands in the infra-red region. Two absorption bands, having their maxima at 7.22μ and 7.88μ respectively, were found by J. Maslakowez [1] for KBr crystals with KNO_3 impurities. He found similar bands for KCl and KI crystals with KNO_3 impurities, and also for KCl and NaCl crystals with KNO_2 and $NaNO_3$ impurities respectively. The presence of a CO_3 anion in the lattice of KCl and KBr crystals gave rise to similar absorption in the region of $7\mu - 8\mu$. Absorption was found in the same region by F. W. Ackermann [2] in the case of AgCl crystals with $AgNO_3$ and $AgNO_2$ impurities.

Now it has been shown by Smakula [3] that the addition of foreign anions to KCl crystals produces an absorption band in the ultra-violet region (for NO_3 at 2000-2200 Å); the absorption of energy causes no luminescent excitation of the crystal.

In this work the absorption in the infra-red region for some ionic crystals with anionic impurities was examined, and characteristic changes produced by ultra-violet illumination were found in the absorption in the infra-red region.

The monocrystals grown by the Kyropoulos method [4] from a melted mass were investigated. The impurities, together with the base substance, were melted in a melting pot. The mixture, from which the crystals were grown, contained no more than 10^{-2} per cent of a mol of the activating substance.

The absorption measurements in the infra-red region were made with a double-beamed Hilger D-209 spectrometer, provided with a NaCl prism. The spectrometer was connected to a recording apparatus. A vacuum Schwarz thermo-couple with a KBr window was used as a detector. The crystals were irradiated with light from a mercury lamp which was placed in one focus of the elliptical mirror. The crystal under examination was placed in the second focus.

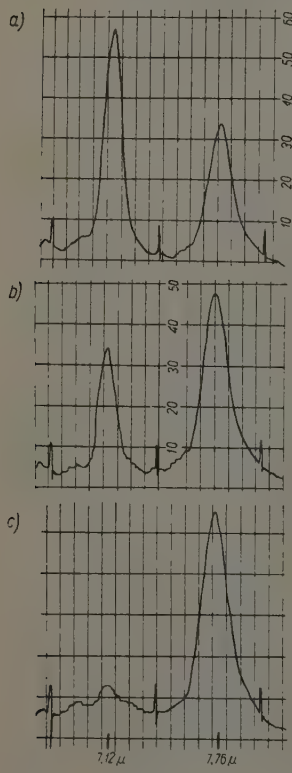

Fig. 1. KBr KNO_3 Crystal
a. before irradiation
b. after 3 min. irradiation
c. after 45 min. irradiation

Two absorption bands were found for KBr crystals with KNO_3 impurities at 7.12 μ and 7.76 μ. The short-wave band has greater intensity; nevertheless, the ratio of intensities in both bands varies and has different values for different crystals (i. e. crystals of different origin). The introduction of CO_3 ions by adding K_2CO_3 gives rise to a 6.9μ band and to 7.12μ and 7.76μ bands which are similar to the ones previously mentioned. The 7.76μ band is of insignificant intensity. The addition of $K_2C_2O_4$ gives rise to similar absorption bands. On introductiong OH cations by adding KOH, 3 bands at 6.7μ, 7.0μ and 7.76μ respectively, appear, the last being of a small intensity. There is also a 7.1μ band concealed under the 7.0μ band.

On irradiating crystals containing all the above impurities with a full light from a mercury lamp, the intensity of the 7.12 μ band was seen to gradually decrease and that of the 7.76 μ band to increase (Figs. 1, 2, 3 and 4). Checking experiments (with illumination through a filter cutting down the ultra-violet light) showed that these changes are produced by the absorption of light in the ultra-violet region. These changes appear to be permanent, but after heating up to a temperature of 650°C. or higher, the band at 7.12 μ is regenerated while at the same time the intensity of the 7.8 μ — band is decreased.

In the case of KI crystals 6.8 μ and 7.12 μ bands are seen to appear. No impurities were added to the material from which the crystals were grown.

The absorption observed was caused by the impurities which were present in the base material. Ultra-violet light caused similar changes as in the case of KBr crystals (viz. disappearance of the band at 7.12 μ, and increase — at 7.8 μ. — Fig. 5).

Irradiation of the crystals under examination with anionic impurities usually caused some coloration, which could be destroyed by illumi-

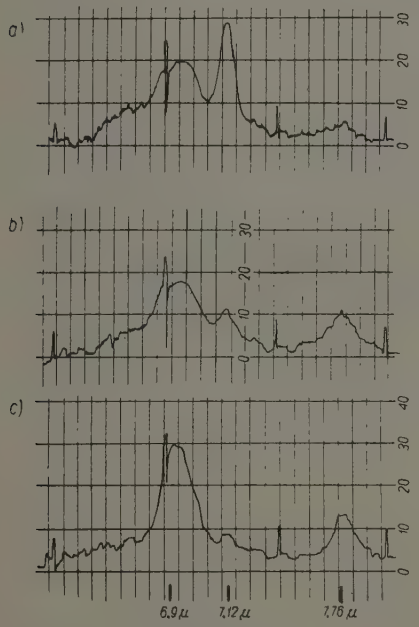

Fig. 2. KBr K_2CO_3 Crystal
a. before irradiation
b. after 3 min. irradiation
c. after 36 min. irradiation

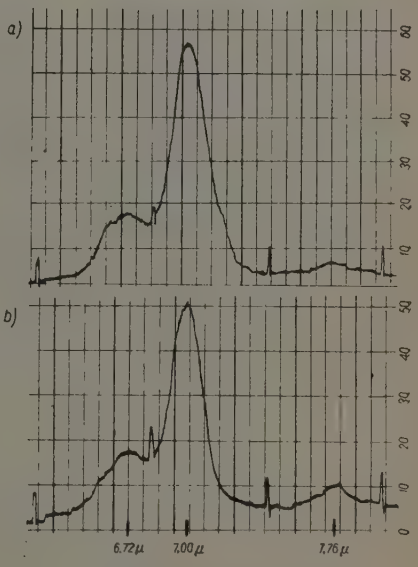

Fig. 3. KBr KOH
a. before irradiation
b. after irradiation

nating the coloured crystals with a white light. Moreover, it was observed that the crystals coloured by irradiation with ultra-violet light and also those decoloured by illumination or heating up to a moderate temperature, were characterized by identical absorption spectra in the infra-red region.

It was also found that there were absorption changes caused by thermal treatment of the crystals. These changes are being further investigated.

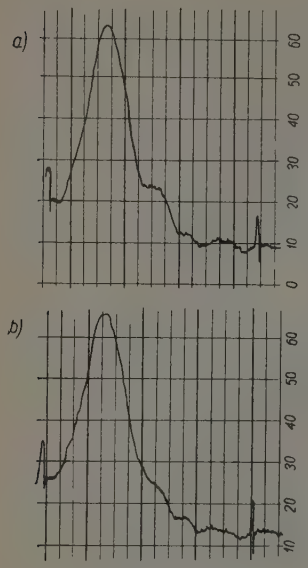

Fig. 4. KBr KOH
a. before irradiation
b. after irradiation

The experimental data for different crystals with various impurities are listed in Table I.

It should be noted that ultra-violet light is responsible in all cases for the appearance or increase of the 7.76 μ — band and for the disappearance of the 7.12 μ — band. The variations in absorption seem to indicate, that the process of introducing into the lattice the anions under investigation is associated with special centres responsible for the absorption in a band of 7.1 μ. These centres are destroyed by ultra-violet light, and at the same time the number of centres responsible for the absorption in the 7.8 μ — band is increased. The latter centres are permanent at room temperature and only at sufficiently high temperatures can they be transformed back into centres responsible for shorter-wave absorption.

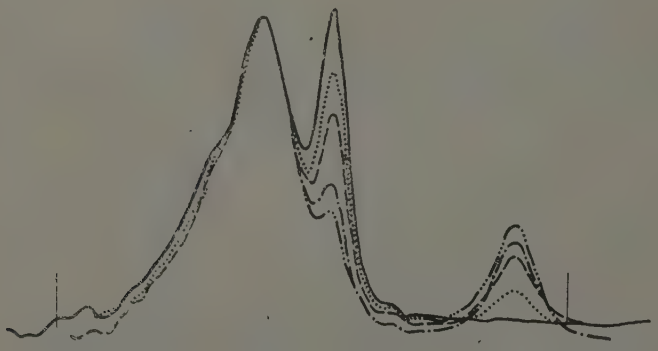

Fig. 5. Absorption of KI crystals before and after irradiation

——— before irradiation
····· after 4 min. irradiation
— — — after 9 min. irradiation
—·—·— after 20 min. irradiation
—···—··· after 40 min. irradiation

Table I

Crystal	Foreign anion	Absorption bands			Explanatory notes
		6.8 μ	7.1 μ	7.8 μ	
KBr	NO_3	−	+	+	before irradiation
		−	traces	increase	after irradiation
	CO_3	+	+	traces	before irradiation
		+	+	increase	after irradiation
	OH	+	+	traces	before irradiation
		+	decrease	increase	after irradiation
KI	?	+	+	−	before irradiation
		+	traces	+	after irradiation

The author is grateful to Professor L. Sosnowski for his interest and help throughout the work. He feels particularly indebted to the late Professor S. Pieńkowski for enabling him to carry out this investigation and for the kindness which he constantly showed him.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. III, No. 3, 1955

CHEMISTRY

On the synthesis and degradation of some derivatives of tetrahydro-1,3-oxazine

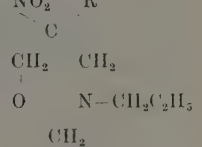
by
D. GÜRNE and T. URBAŃSKI

Communicated by T. URBAŃSKI at the meeting of December 13, 1954

It has been shown ([1], [3]) that primary nitroparaffins, when reacting with formaldehyde and ammonia or with primary amines, can yield derivatives of tetrahydro-1,3-oxazine.

The experiments described in the present paper were carried out in order to furnish new proof of the structure of tetrahydro-1,3-oxazine derivatives, by their *step-wise degradation.*

By using a homologous series of nitroparaffins from nitromethane to 1-nitro-n-butane and formaldehyde, and benzylamine, a number of new compounds of the general formula (I) were obtained: all of them were 5-nitro-5-alkyl-3-benzyl-tetrahydro-1,3-oxazines.

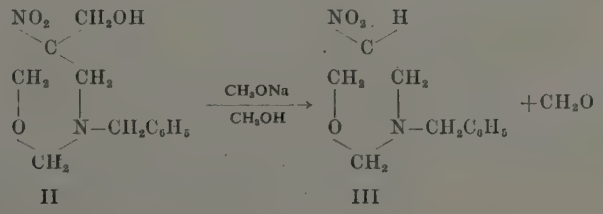


I

(a) R CH_3
(b) R $= C_2H_5$
(c) R C_3H_7

The compounds (Ia), (Ib) and (Ic) were prepared from nitroethane, 1-nitropropane and 1-nitro-n-butane respectively. (Although Senkus [2] can lay claim to having prepared the compounds (Ia), (Ib) and (Ic), he did not furnish experimental details and did not describe any of the physical properties of these products).

Two products described previously by the authors of the present paper [3], were prepared from nitromethane: the compounds (III) and (II). The latter when acted on by sodium methoxyde lost a molecule of formaldehyde and yielded the compound (III):



II III

All compounds: (Ia,b,c), (II), (III) were subjected to degradation by the following method.

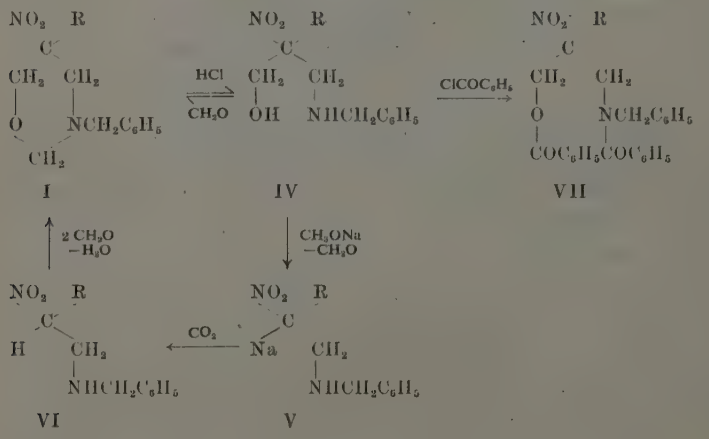
(1) By heating the ring-compounds with hydrochloric acid, a molecule of formaldehyde was split off and aminoalcohols of the open chain-structure (IV) were obtained. An analogous reaction has been described in former papers [1]. It was found that a considerable shortening of the time necessary to the ring-opening, e. g. from 15 hours to 5 hours, could be achieved, when a solution in concentrated hydrochloric acid was irradiated with ultraviolet light during heating on a steam-bath.

However, the best results as regards ring-opening can be obtained by using hydrochloric acid diluted with alcohol to 1% HCl (this technique has been described by Burke [4]) to open the hetero-ring of benzoxazine derivatives.

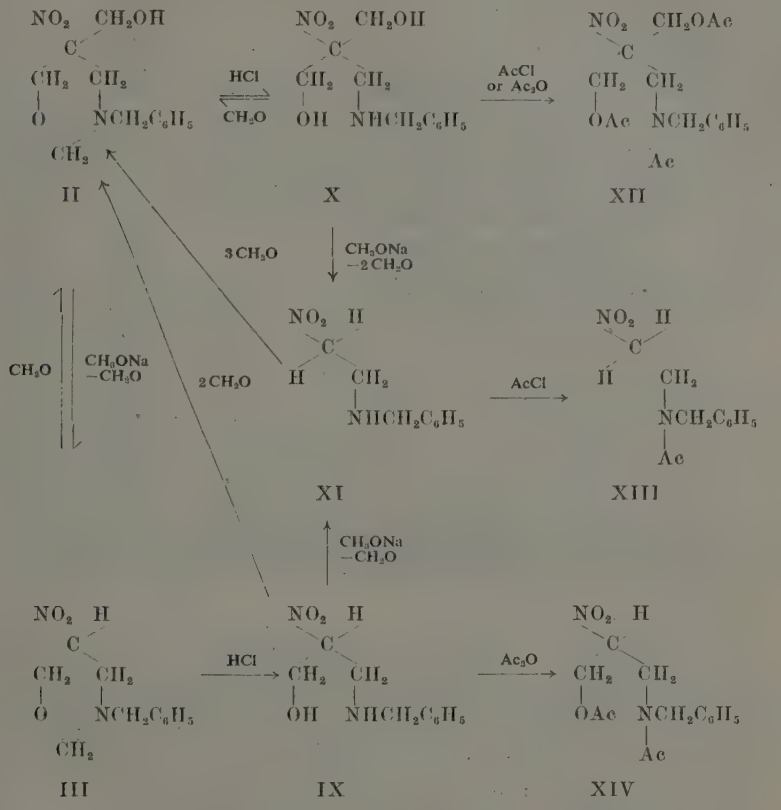
The aminoalcohols prepared according to this method, when warmed with formaldehyde, furnished the starting products of the tetrahydro-1,3-oxazine type.

(2) By treating aminoalcohols (IV) with sodium methoxyde in methanol-solution a mol of formaldehyde was cleaved off and sodium salts (V) of aminonitrocompounds resulted. These, under the action of carbon dioxide, furnished aminonitrocompounds (VI), which upon reaction with formaldehyde in the presence of sodium-hydrogen carbonate, at pH 7.5—8.0 furnished the initial substances (I).

The structure of compounds (IV) was established: (1) by analysis, (2) by the formation on N-nitrosocompounds, which on being warmed with conc. HCl again yielded the amines (IV), and (3) by benzoylation, which yielded products (VII). In a similar way, (VI) when acted on by nitrous acid yielded the oily N-nitrosocompounds and when acted on by tosyl chloride — the N-tosyl derivatives.



When the products from nitromethane (II) and (III) were subjected to degradation, similar reactions were obtained:



The products (II) and (III) when warmed with hydrochloric acid diluted with alcohol to 1% furnished the aminoalcohols (IX) and (X) respectively.

Both aminoalcohols (IX) and (X), under the action of sodium methoxide lost formaldehyde, and furnished the same product (XI). The structure of (IX), (X) and (XI) was established on the basis: (a) of the analytical results, (b) of the reaction with nitrous acid, furnishing oily N-nitrosocompounds, which when warmed with hydrochloric acid yielded again the substances (IX), (X) and (XI) respectively, (c) of the reaction of (X) and (XI) with acetyl chloride, which furnished the acetyl derivatives (XII) and (XIII).

Appended is a list of the new compounds described in the present paper, together with their melting points.

Name	Number	m. p.
5-nitro-5-methyl-3-benzyl-tetrahydro-1,3-oxazine	I a	66– 68°
5-nitro-5-ethyl-3-benzyl-tetrahydro-1,3-oxazine	I b	68– 70°
5-nitro-5-propyl-3-benzyl-tetrahydro-1,3-oxazine	I c	46– 48°
5-nitro-3-benzyl-tetrahydro-1,3-oxazine	II	44– 46°
5-nitro-5-(hydroxymethyl)-3-benzyl-tetrahydro-1,3-oxazine	III	140–142°
N-[(2-nitro-2-hydroxymethyl)-propyl]-benzylamine hydrochloride	IV a	102–104°
N-[(2-nitro-2-hydroxymethyl)-butyl]-benzylamine hydrochloride	IV b	150–152°
N-[(2-nitro-2-hydroxymethyl)-pentyl]-benzylamine hydrochloride	IV c	136–138°
N-[(2-nitro-2-hydroxymethyl)-ethyl]-benzylamine hydrochloride	IX	150° dec.
N-[(2-nitro-2,2-dihydroxymethyl)-ethyl]-benzylamine hydrochloride	X	177° dec.
N-(2-nitro-propyl)-benzylamine hydrochloride	VI a	148–150°
N-(2-nitro-butyl)-benzylamine hydrochloride	VI b	150–151°
N-(2-nitro-pentyl)-benzylamine hydrochloride	VI c	152–154°
N-(2-nitro-ethyl)-benzylamine hydrochloride	XI	147° dec.
N, O-dibenzoyl derivative of IV a	VII a	112–114°
„ „ „ „ IV b	VII b	105–107°
„ „ „ „ IV c	VII c	90– 92°
N, O, O-triacetyl derivative of X	XII	98–100°
N-tosyl derivative of VI a	VIII a	82– 84°
„ „ „ „ VI b	VIII b	88– 89°
„ „ „ „ VI c	VIII c	110–112°
N-acetyl derivative of XI	XIII	193–195°

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. III, No. 3, 1955

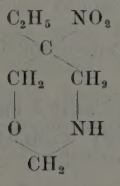
CHEMISTRY

On some properties of tetrahydro-1,3-oxazines deriving from 1-nitrobutane or 1-nitro(iso)butane

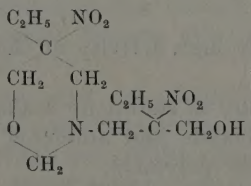
by
T. URBAŃSKI, J. KOLESIŃSKA and H. PIOTROWSKA

Communicated by T. URBAŃSKI at the meeting of December 13, 1954

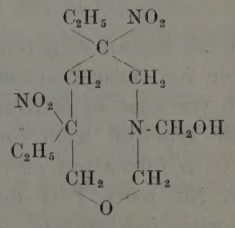
It has been shown by E. L. Hirst, J. K. N. Jones, S. Minahan, F. W. Ochyński, A. T. Thomas and T. Urbański [1], that 1-nitropropane can react with formaldehyde and ammonia yielding three new ring compounds. Two of them (I and II) are derivatives of tetrahydro-1,3-oxazine, and one is a derivative of 1-oxa-3-azacyclooctane (III) The compounds which contain these two rings belong to ones which are relatively little known.



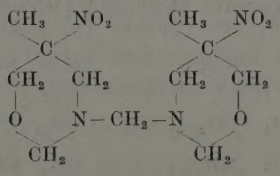
I



II



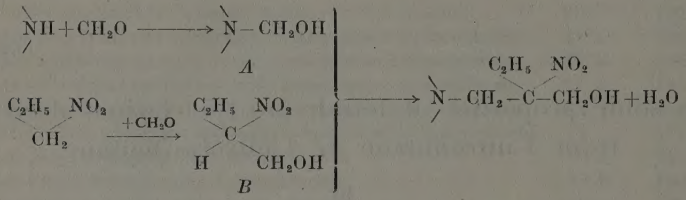
III



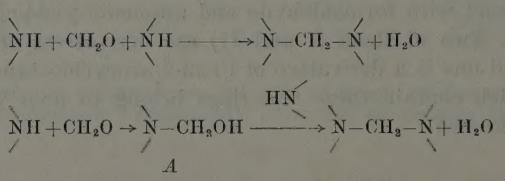
IV

According to T. Urbański and E. Lipska [2], a new compound (IV) with two tetrahydro-1,3-oxazine rings is formed as a result of the reaction of nitroethane formaldehyde and ammonia.

Compound (II) is an N-derivative of (I). The former can be obtained by the action on (I) of 1-nitropropane and two mols of formaldehyde. The reaction giving rise to (I) may occur through the formation of the intermediates (*A*) and (*B*), according to the following diagram:



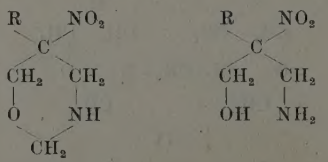
In the case also of a 1-oxa-3-azacyclooctane derivative, the activity of N-hydrogen produced the N-hydroxymethyl compound (III). Finally, in the instance of nitroethane, two tetrahydro-1,3-oxazine rings combined together through a molecule of formaldehyde, probably as shown in the following diagram:



In all instances a relatively high activity of the N-hydrogens is responsible for these reactions.

In the case of nitromethane, as shown by Malinowski and Urbański [3], the reaction with formaldehyde and ammonia is very violent and leads to a formation of resinous polymers.

In the work here described, the authors found that 1-nitrobutane and 1-nitro-(iso)-butane can readily form tetrahydro-1,3-oxazine ring compounds of one simple type (V) only: 5-nitro-5-n-propyl (and 5-iso-propyl)-tetrahydro-1,3-oxazines (Va) and (Vb) respectively



V VI

(a) R = n-C_3H_7 (a) R = n-C_3H_7

(b) R = i-C_3H_7 (b) R = i-C_3H_7

and the open chain aminoalcohols:

2-nitro-2-hydroxymethylpentylamine (VIa)

and 2-nitro-2-hydroxymethyl-3-methylbutylamine (VIb).

All attempts to combine further products (V) with formaldehyde through the ring-nitrogen failed to produce any change in the compound. This would suggest a relatively low activity of the N-hydrogens of both compounds (Va) and (Vb).

Both compounds were prepared either from 1-nitrobutanes or from the products of the addition of two mols of formaldehyde to the nitroparaffins, i. e. from 2-nitro-2-n-propyl-propanediol-1,3 and 2-nitro-2-isopropyl-propanediol-1,3. The preparation consisted in warming 1 mol of either nitrobutane with 3 mols of formaldehyde and 1 mol of ammonia, or 1 mol of either diol with 1 mol of formaldehyde and 1 mol of ammonia.

After several hours of heating on a steam-bath an oily product resulted. This was dried and dissolved in alcoholic hydrochloride. Crystalline hydrochlorides of (Va) or (Vb) precipitated. A higher yield (10—12% of the theoretical) was obtained when diols, instead of paraffins, were used as starting substances.

In addition to the products (V), aminoalcohols (VI) in the form of more soluble hydrochlorides were also obtained and separated from (V) by fractional crystallisation.

Both hydrochlorides (Va) and (Vb) yielded oily N-nitrosoderivatives, which on being warmed with hydrochloric acid led to the initial hydrochlorides. The free bases (Va) and (Vb) reacted with methyl iodide to yield dimethyl iodides.

After being boiled with conc. hydrochloric acid for several hours, the hydrochlorides (Va) and (Vb) lost one mol of formaldehyde and formed the hydrochlorides (VIa) and (VIb) respectively. The free bases (VI), when warmed with formaldehyde underwent cyclisation, resulting in bases (V). More details will be given in two other papers [4].

The melting points of the compounds described in this paper are collected in the table on page 182.

According to S. Ślopek [4], tetrahydro-1,3-oxazine derivatives (I) and (V) show a definite bacteriostatic action *in vitro* against various *Mycobateria*. Bacteriostatic concentrations were found to be:

(I) (designated as *T* 41) 7.5—125 mg. %

(Va) (designated as *T* 176) 62.5—125 mg. %.

The product *T* 41 was examined by S. Ślopek *in vivo* against experimental tuberculosis in mice, and an action similar to that of streptomycine was observed [5].

TABLE

Products	m. p.
5-nitro-5-propyl-tetrahydro-1,3-oxazine hydrochlorides:	
(V a) R = n–C_3H_7	190–192°
(V b) R = i–C_3H_7	190°
N-dimethyl iodide of (V a)	199–200°
(V b)	201–202°
(VI a) R = n–C_3H_7	169–170°
(VI b) R = i–C_3H_7	165–167°
Picrate (V a)	163–164°
(V b)	167–168°
(VI b)	158–160°
O-N-dibenzoyl derivative of (VI a)	101–102°

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[1] E. L. Hirst, J. K. N. Jones, S. Minahan, F. W. Ochyński, A. T. Thomas, T. Urbański, J. Chem. Soc. (1947), 924.
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